Department of Mechanical and Production Engineering
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Lecture Notes

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Lecture 1: Introduction to Fluid Mechanics August 25, 2025

2 Fundamental Concepts

2.1 Fluid as a Continuum

Fluids are normally experienced as being continuous or "smooth" when perceived in the macroscopic world. If one looks at a fluid microscopically one would start to see that the fluid is not continuous at all but instead composed of distinct particles. We wish to determine the minimum volume, $\partial V'$ that a point C must be such that we can talk about the fluid being continuous at this point. In other words, under what circumstances can a fluid be treated as a continuum, for which, by definition, properties vary smoothly over all points.

We define the mass ∂m as the instantaneous number of molecules in ∂V and the mass of each of these molecules. The average density inside volume ∂V is hence given by $\rho = \frac{\partial m}{\partial V}$. It is important to note that this necessarily is an *average value* as the number of molecules in ∂V and hence the density fluctuates. Due to the law of large numbers one will experience that for very small volume the density will fluctuate greatly over time, however at a certain volume $\partial V'$, the density becomes stable and will not fluctuate greatly over time. For $\partial V = 0{,}001\,\mathrm{mm}^3$ (about the size of a grain of sand) there will already be an average of $2{,}5 \cdot 10^{13}$ molecules present. Hence a liquid can be treated as a continuous medium as long as we consider a "point" to be no smaller than about this size – at least this is sufficiently precise for most engineering applications.

This continuum hypothesis is an integral part of fluid mechanics. It is valid when treating the behaviour of fluids under normal conditions and only breaks down when the mean free path of the molecules becomes the same order of magnitude as the smallest characteristic dimension of the problem.

As a consequence of the continuum assumption, each property of the fluid is assumed to have a definite value at each point in space. I.e. density, temperature, velocity, and so on are each continuous functions of both position and time. We now have a definition of the density at a point:

$$\rho \equiv \lim_{\partial V \to \partial V'} \frac{\partial m}{\partial V}.$$

As point C was chosen arbitrarily the density at any other point in the liquid can be determined in the same manner. If one measures this simultaneously for all points in the fluid, an expression for the density distribution as a function of the space coordinates $\rho = \rho(x, y, z)$ can be found.

The density at a specific point may also vary with time. Therefore the complete representation of density, the so-called *field representation*, can be written as:

$$\rho = \rho(x, y, z, t) \tag{1}$$

As density is a scalar quantity the field given by Equation (1) is a scalar field.

The density can also be expressed as the specific gravity, i.e. the weight compared to the maximum density of water, $\rho_{\mathrm{H_2O}} = 1000\,\frac{\mathrm{kg}}{\mathrm{m}^3}$ at $4\,°\mathrm{C}$. Thus the specific gravity of a substance can be found as:

$$\mathrm{SG} = \frac{\rho}{\rho_{\mathrm{H_2O}}}.$$

Another useful property is the *specific weight*, γ, of a substance. It is defined as the weight of a substance per unit volume, i.e.:

$$\gamma = \frac{mg}{V} \implies \gamma = \rho g.$$

2.2 Velocity field

A very important property defined by a field is the velocity field, given by:

$$\mathbf{V} = \mathbf{V}(x, y, z, t) \tag{2}$$

As velocity is a vector quantity, the field given in Equation (2) is a vector field.

The velocity vector, **V**, can also be written in terms of its scalar components. Denoting the components in the x-, y-, and z-directions by u, v, and w, respectively we get:

$$\mathbf{V} = u\hat{i} + v\hat{j} + w\hat{k}.$$

Here, each component, u, v, and w, will generally be functions of x, y, z and t.

We also need to be make sure to remember that $\mathbf{V}(x, y, z, t)$ represents the velocity of a fluid particle passing through the point (x, y, z) at time t. Therefore $\mathbf{V}(x, y, z, t)$ should be thought of as the velocity field of the entire fluid and not of an individual particle.

If properties at every point in a flow field are constant with respect to time, the flow is termed *steady*. This is defined mathematically as:

$$\frac{\partial \eta}{\partial t} = 0.$$

Where η is any fluid property. Hence, for steady flow:

$$\frac{\partial \rho}{\partial t} = 0 \text{ or } \rho = \rho(x, y, z)$$

and

$$\frac{\partial \mathbf{V}}{\partial t} = 0 \text{ or } \mathbf{V} = \mathbf{V}(x, y, z).$$

As such any property may vary from point to point in the field but remain constant with time at every point for steady flow.

2.3 One-, Two-, and Three-Dimensional flows

A flow is either one-, two-, or three-dimensional depending on the amount of spatial coordinates required to specify the velocity field. Equation (2) shows that the velocity field in some cases is a function of three spatial coordinates and time – in this case it is a three-dimensional flow.

Almost all flows are three-dimensional in nature – however, analysis based on fewer dimensions is often sufficient. The complexity of analysis increases sharply as more dimensions are added, and oftentimes in engineering a one-dimensional analysis is sufficient.

To not break the continuum assumption all fluids must have zero velocity at any solid surface, e.g. the inner side of a pipe. Due to this all flow in pipes is inherently three-dimensional. To simplify the analysis one often uses the notion of *uniform flow* at a given cross-section. In a flow that is uniform at a given cross-section, the velocity is constant across any section normal to the flow as shown in Figure 2.1. Under this assumption, the flow simplifies to be simply a function of x alone. The term *uniform flow field* is used to describe a flow in which the velocity is constant throughout the entire field.

Definition 1: Uniform flow at a cross section

An assumption that states that a fluid has the same velocity everywhere in a given cross section as shown on Figure 2.1. This actually breaks the continuum hypothesis but it makes a lot of calculations easier.

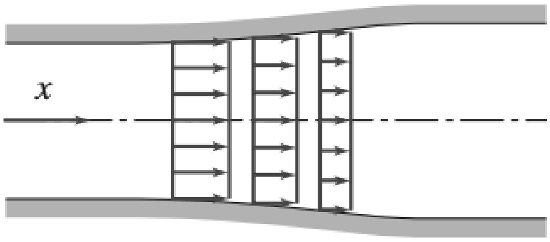


Figure 2.1: *Uniform flow* at a given cross-section.

Definition 2: Uniform flow field

A flow field where the velocity is constant everywhere throughout the flow field.

2.4 Timelines, Pathlines, Streaklines, and Streamlines

Wind tunnels have traditionally often utilized to visualize flow fields. In modern times the advent of computer simulations has meant that these have become much more prevalent recently. A few different terms must be defined for proper understanding of these.

Definition 3: Timeline

A *timeline* is produced by marking adjacent fluid particles in a flow field at a given instant. Subsequent observation of the timeline can give insights into the flow field.

Definition 4: Pathline

A *pathline* is the trajectory traced out by a moving fluid particle. These can be visualized by marking a fluid particle at a given instant, e.g. with dye or smoke, and then tracing the path of this particle as it moves through the field.

Definition 5: Streakline

A *streakline* is the line traced out by marking the fluid particles at a fixed point in space, e.g. with smoke or dye. This is therefore a way to see how fluid particles that passed through a specific point behave afterwards.

Definition 6: Streamlines

A *streamline* are lines drawn in a flow field such that at any given instant they are tangent to the velocity vector at every point in the flow. This means there can be no flow across a streamline. These are the most commonly used visualization technique

We can use the velocity field to derive the shapes of streaklines, pathlines and streamlines. As the streamlines are parallel to the velocity vector, for a two dimensional flow field, we can write:

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{\text{streamline}} = \frac{v(x,y)}{u(x,y)} \tag{3}$$

Note that these are obtained at a given instant in time. If the flow is unsteady, time t is held constant in Equation (3). Solution of the equation gives $y = y(x)$, with an undetermined integration constant, the value of which depends on the particular streamline.

For pathlines, we let $x = x_p(t)$ and $y = y_p(t)$ where $x_p(t)$ and $y_p(t)$ are the instantaneous coordinates of a specific particle. In this case we get

$$\left.\frac{\mathrm{d}x}{\mathrm{d}t}\right|_{\text{particle}} = u(x,y,t) \qquad \left.\frac{\mathrm{d}y}{\mathrm{d}t}\right|_{\text{particle}} = v(x,y,t) \tag{4}$$

The simultaneous solution of these equations gives the path of a particle in parametric form, $x_p(t), y_p(t)$.

For streaklines, the first step is to compute the pathline of a particle with Equation (4) that was released from the streak source at x_0, y_0 at time t_0, in the form

$$x_{\text{particle}}(t) = x(t, x_0, y_0, t_0) \qquad y_{\text{particle}}(t) = y(t, x_0, y_0, t_0).$$

Then now, instead of interpreting this as the position of a particle over time, we instead write the equations as:

$$x_{\text{streakline}}(t_0) = x\left(t, x_0, y_0, t_0\right) \qquad y_{\text{streakline}}(t_0) = y\left(t, x_0, y_0, t_0\right) \tag{5}$$

Equation (5) gives the line generated (by time t) from a streak source placed at (x_0, y_0). In these equations t_0 is varied from 0 to t to show the *instantaneous* positions of all particles released up to time t.

2.5 Stress Field

To understand the behaviour of fluids one must first understand the nature of the forces that act upon fluid particles. A fluid particle can experience either:

- *Surface forces*, e.g. pressure or friction, that are generated due to contact with other particles or surfaces
- *Body forces*, e.g. gravity and electromagnetic, that are experienced throughout the particle.

Surface forces on a fluid particle leads to *stresses*. Stress is an important concept when describing how forces acting on the boundaries of a medium are transmitted throughout the medium.

We consider the surface of a particle in contact with other fluid particles and the contact force being generated between these. Let $\delta\mathbf{A}$ be a portion of the surface at some point C. The orientation of $\delta\mathbf{A}$ is given by the unit vector $\hat{n}$, which is perpendicular to the surface as seen in Figure 2.2.

The force, $\delta\mathbf{F}$, acting on the surface portion $\delta\mathbf{A}$ may be split into two components – a *normal stress* σ_n normal to the surface and a *shear stress* τ_n tangential to the surface, defined as:

$$\sigma_n = \lim_{\sigma A_n \to 0} \frac{\delta F_n}{\delta A_n}$$
$$\tau_n = \lim_{\delta A_n \to 0} \frac{\delta F_t}{\delta A_n}.$$

The subscript n on the stress is a reminder that the stresses are associated with the surface portion $\delta\mathbf{A}$ through C, which has an outward normal in the $\hat{n}$ direction.

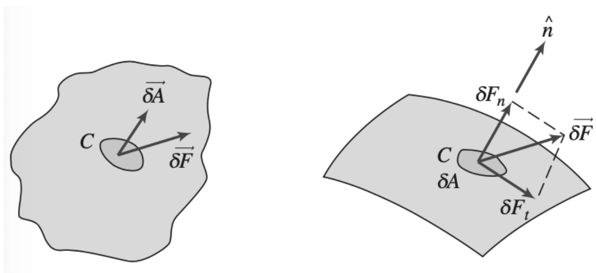
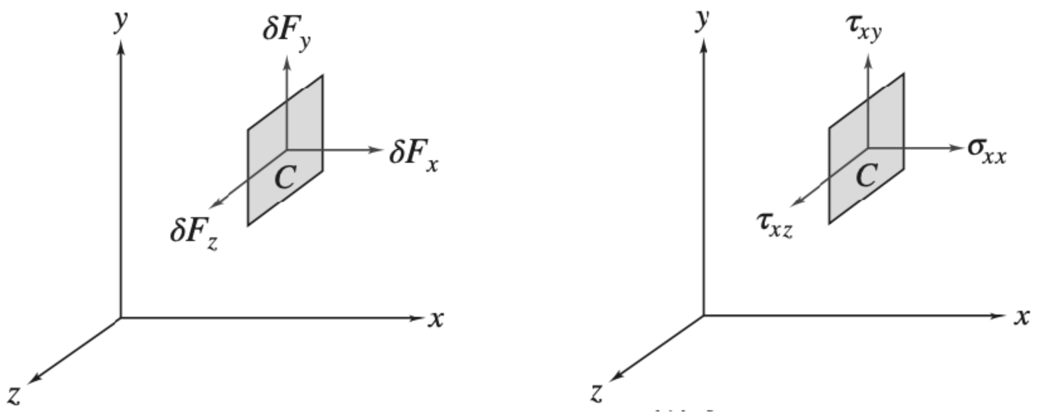


Figure 2.2: Stress in a continuum.

We consider the stress on the element δA_x whose normal is in the $+x$-direction. We can then split the force acting upon this point $\delta\mathbf{F}$ into components along each coordinate direction. By dividing the magnitude of each force component by the area δA_x and taking the limit as δA_x approaches zero we define three stress components as:

$$\sigma_{xx} = \lim_{\delta A_x \to 0} \frac{\delta F_x}{\delta A_x}$$
$$\tau_{xy} = \lim_{\delta A_x \to 0} \frac{\delta F_y}{\delta A_x}$$
$$\tau_{xz} = \lim_{\delta A_x \to 0} \frac{\delta F_z}{\delta A_x}.$$

This is also shown graphically in Figure 2.3.



(a) Force components (b) Stress components

Figure 2.3: Force and stresses at surface element δA_x.

Here the first subscript (x) indicates the *plane* on which the stress acts, in this case a surface perpendicular to the x-axis. The second direction indicates the *direction* in which the stress acts. I.e. consideration of the element δA_y would lead to the stresses σ_{yy}, τ_{yx} and τ_{yz} and similarly for δA_z.

As the coordinate system was chosen arbitrarily it is easily realized that one can define an infinite amount of stresses through a point C depending on how the axes are placed. Luckily, the state of stress at any point is completely described by the stresses acting in any three mutually perpendicular planes through the point. Therefore the stress at a point is specified by the nine components:

$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix}.$$

Planes are normally named for the direction in which their normal vector is pointing. Also we normally define a stress component to be positive when the stress component and the plane on which it acts are either both positive or negative.

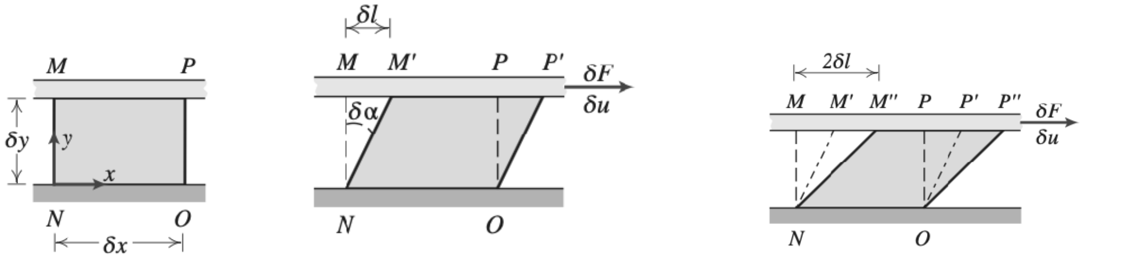
2.6 Viscosity

For a solid stresses develop when the material is elastically deformed or strained; for a fluid, shear stresses instead arise due to viscosity. For a fluid at rest there will be no shear stresses.

We consider the behaviour of a fluid element between two infinite planes sketched on Figure 2.4. The rectangular fluid element is initially at rest at time t. We now suppose a constant rightward force δF_x is applied to the upper plate such that it is dragged across the fluid at constant velocity δu. The shearing action of the plates produces a shear stress τ_{yx}, which acts on the fluid element, and is given by:

$$\tau_{yx} = \lim_{\delta A_y \to 0} \frac{\delta F_x}{\delta A_y} = \frac{\mathrm{d}F_x}{\mathrm{d}A_y}$$

where δA_y is the contact area between the fluid element between the plade and δF_x is the force exerted by the plate on the element.



(a) Fluid element at time t, (b) Deformation of fluid element at time $t+\delta t$, (c) Deformation of fluid element at time $t+2\delta t$.

Figure 2.4: Deformation of a fluid element.

Focusing on the time interval δt (Figure 2.4b) the deformation of the fluid is given by:

$$\text{deformation rate} = \lim_{\delta t \to 0} \frac{\delta \alpha}{\delta t} = \frac{\mathrm{d}\alpha}{\mathrm{d}t}.$$

We will now seek to express $\frac{\mathrm{d}\alpha}{\mathrm{d}t}$ in terms of measurable quantities. The distance, δl, between the points M and M' is given by:

$$\delta l = \delta u \delta t.$$

For small angles this simplifies to:

$$\delta l = \delta y \delta \alpha.$$

Equating these two expressions gives:

$$\frac{\delta \alpha}{\delta t} = \frac{\delta u}{\delta y}.$$

By taking the limits of both sides of this, we get:

$$\frac{\mathrm{d}\alpha}{\mathrm{d}t} = \frac{\mathrm{d}u}{\mathrm{d}y}.$$

Thus, the fluid element will, when subjected to a shear stress τ_{yx}, experience a rate of deformation given by $\frac{\mathrm{d}u}{\mathrm{d}y}$. We have thus established that any fluid will flow and have a shear rate when subjected to a shear stress. Fluids in which the shear stress is proportional to the deformation rate are termed *Newtonian* and fluids for which this is not the case are termed *non-Newtonian*.

2.7 Newtonian Fluid

Most common fluids are Newtonian under normal conditions. If the fluid on Figure 2.4 is Newtonian, then:

$$\tau_{yx} \propto \frac{\mathrm{d}u}{\mathrm{d}y}.$$

The constant of proportionality between these two quantities is the *absolute viscosity* also called the dynamic viscosity, μ. Thus in terms of the coordinates on Figure 2.4 Newton's law of viscosity for one-dimensional flow is:

$$\tau_{yx} = \mu \frac{\mathrm{d}u}{\mathrm{d}y} \tag{6}$$

By dimensional analysis on Equation (6) it can be seen that the units of viscosity are kg / m s or Pa s.

In fluid mechanics the ratio of absolute viscosity μ to density ρ often arises. This ratio is called the *kinematic viscosity* and is represented by ν. The units for this is 1 stoke ≡ 1 cm^2 / s.

2.8 Non-Newtonian Fluids

Many empirical equations have been proposed to model the observed relations between τ_{yx} and $\frac{\mathrm{d}u}{\mathrm{d}y}$ for time-independent non-Newtonian fluids. For many engineering applications the power law model is sufficient, for which one-dimensional flow becomes:

$$\tau_{yx} = k\left(\frac{\mathrm{d}u}{\mathrm{d}y}\right)^n.$$

Here n is called the *flow behaviour index* and k is called the *consistency index*. Newton's law of viscosity is a special case of this with $n = 1$ and $k = \mu$. To ensure that τ_{yx} has the same sign as $\frac{\mathrm{d}u}{\mathrm{d}y}$ the equation is rewritten as:

$$\tau_{yx} = k\left|\frac{\mathrm{d}u}{\mathrm{d}y}\right|^{n-1}\frac{\mathrm{d}u}{\mathrm{d}y} = \eta\frac{\mathrm{d}u}{\mathrm{d}y} \tag{7}$$

Here the term $\eta = k\left|\frac{\mathrm{d}u}{\mathrm{d}y}\right|^{n-1}$ is referred to as the *apparent viscosity*. When using Equation (7) we end up with a viscosity η that is used in a formula in the same form as Equation (6). The big difference between these two is that while μ is constant (at a constant temperature), η depends on the shear rate.

Fluids for which the apparent viscosity decreases with increasing deformation rate ($n < 1$) are called *pseudoplastic* or shear thinning fluids. Most non-Newtonian fluids are of this type. If the apparent viscosity increases with deformation rate ($n > 1$) the fluid is termed *dilatant* or shear thickening.

A "fluid" that behaves as a solid until a minimum yield stress τ_y is exceeded and subsequently exhibits a linear relation between stress rate and deformation is termed an ideal or *Bingham plastic*. The shear stress model for a Bingham plastic is:

$$\tau_{yx} = \tau_y + \mu_p\frac{\mathrm{d}u}{\mathrm{d}y}.$$

The apparent viscosity may also be time-dependent. *Thixotropic* fluids show a decrease in η with time at a constant stress and *Rheopectic* fluids show an increase in η with time. Some fluids partially return to their original shape when the stress is removed – these are called *viscoelastic*.

2.9 Surface Tension

Droplets can either "flatten out" or remain as little drops when dropped on a surface. We define a liquid as *wetting* a surface when the contact angle is $< 90°$. This is shown on Figure 2.5.

Surface tension arises at the interface between the liquid and a solid – this interface acts like a stretched elastic membrane in turn creating surface tension. This membrane is completely described by two features: the contact angle θ and the magnitude of the surface tension σ [N/m]. Both of these depend on both the type of liquid and the characteristics of the surface with which it shares an interface.

(a) A "wetted" surface (b) A nonwetted surface

Figure 2.5: Surface tension effects on water droplets.

2.10 Description and Classification of Fluid Motions

The two most difficult aspects of a fluid mechanics analysis to deal with are: (1) the fluid s viscous nature and (2) its compressibility. When fluid mechanics first was developed it was occupied with a frictionless and incompressible fluid. Whilst extremely elegant this leads to the infamous result called d'Alembert's paradox, which states that all bodies experience zero drag as they move through a liquid – which of course is not consistent with observations.

2.10.1 Viscous and Inviscid Flows

Any object moving through a fluid will experience gravity and an aerodynamic drag force. This drag force is in part due to viscous friction and in part due to pressure differences being produced as the liquid moves out of the way of the object. We can estimate whether or not viscous forces are negligible compared to pressure forces by computing the Reynolds number:

$$\mathrm{Re} = \rho \frac{VL}{\mu}.$$

where ρ and μ are the density and viscosity of the fluid, respectively, and V and L are the "characteristic" velocity and size scale of the flow, respectively. If the Reynolds number is "large" viscous effects will be small; however, if the Reynolds number is neither large nor small, both are important.

The idealized notion of frictionless flow is called *inviscid flow*. It predicts streamlines as shown in Figure 2.6a. These streamlines are symmetric front to back. As flow between two streamlines is constant the velocity in the vicinity of points A and C must be relatively low compared to the velocity at point B. Hence, points A and C have large pressures whereas B will be a point of low pressure. In fact, the pressure distribution around the sphere is symmetric and there is therefore no net drag force due to pressure. As we are assuming inviscid flow there will be no drag force due to friction either – this is the d'Alembert paradox of 1752.

(a) Inviscid flow (b) Viscous flow

Figure 2.6: Incompressible flow over a sphere.

The answer to this was obtained by Prandtl in 1904. The no-slip condition requires that the velocity everywhere at the surface of the ball must be zero, but inviscid theory states that it is high at point B. Prandtl suggested that even though friction is negligible for high-Reynolds number flows, there will always be a thin boundary layer, in which friction is significant and over which the velocity will increase rapidly from zero at the surface to the value predicted by inviscid theory at the edge of the boundary layer.

This thin boundary layer explains why drag arises. The boundary layer however also has another important consequence. It often leads to bodies moving through a fluid having a *wake* as shown on Figure 2.6b. Here point D is termed a *separation point*, where fluid particles are pushed of the object thus creating the wake. It turns out that this wake always will have low pressure compared to the front of the sphere, hence the sphere now experiences quite a large *pressure* or *form drag*.

We can now also begin to see how *streamlining* works. The main drag force in most aerodynamics is due to the low pressure wake. If we can reduce or even eliminate this wake the drag will be greatly reduced. Imagine that the sphere was instead teardrop shaped – the pressure gradient will not be changing as quickly along the back half of the object and thus the wake will be smaller leading to less drag. This illustrates the *very* significant different between inviscid flow ($\mu = 0$) and flows where the viscosity can be assumed negligible but not zero ($\mu \to 0$).

2.10.2 Laminar and Turbulent flows

A faucet turned on at a very low flow rate will lead to water running out smoothly – if you increase the flow rate the water will exit in a much more chaotic manner. In fluid dynamics these are termed either *laminar flow* or *turbulent flow*. Oftentimes turbulence is unwanted but unavoidable.

The velocity of laminar flow is simply u. The velocity of turbulent flow is given by the mean velocity $\overline{u}$ plus the three components of randomly fluctuating velocity u', v', and w'. Many turbulent flows may be steady in the mean, but the presence of these random velocity fluctuations makes analysis of turbulent flows extremely difficult. In one-dimensional laminar flow, the shear stress is related to the velocity gradient by the simple relation:

$$\tau_{yx} = \mu \frac{\mathrm{d}u}{\mathrm{d}y}.$$

For a turbulent flow, no such simple relation is valid. In turbulent flow momentum can be transported across streamlines and therefore there us no universal relationship between the stress field and the mean-velocity field. This means we often have to rely on semi-empirical theories for analyzing turbulent flow.

2.10.3 Compressible and Incompressible Flows

A flow in which the variation in density is negligible is termed *incompressible* and those in which density variations are not negligible are called *compressible*. Gasses are normally treated as compressible, whereas liquids are normally treated as incompressible. At high temperatures, however, compressibility effects can start to become important. Pressure and density changes in liquids are related by the *bulk compressibility modulus* or the modulus of elasticity,

$$E_v \equiv \frac{\mathrm{d}p}{\frac{\mathrm{d}\rho}{\rho}}.$$

If the bulk modulus is independent of temperature, then density is only a function of pressure (the fluid is said to be *barotropic*).

Gas flows with negligible heat transfer can be considered incompressible so long as the flow speeds are small relative to the speed of sound. The ratio of flow speed, V, to the local speed of sound, c, in the gas is defined as the Mach number:

$$M \equiv \frac{V}{c}.$$

For $M < 0{,}3$ the maximum variation in density is less than 5%. Thus gas flows with $M < 0{,}3$ can be treated as being incompressible.

2.10.4 Internal and External Flows

Flows completely bounded by solid surfaces are called either *internal*, *pipe*, or *duct flows*. Flows over bodies immersed in an unbounded fluid are termed *external flows*.

An example of an internal flow is that of water in a pipe. The Reynolds number for pipe flows is defined as $\mathrm{Re} = \rho \overline{V} D/\mu$, where $\overline{V}$ is the average flow velocity and D is the pipe diameter. Based on this one can predict if the flow will be laminar or turbulent. For $\mathrm{Re} < 2300$ the flow will generally by laminar and turbulent for larger values. Flow in a pipe of constant diameter will always be entirely laminar or entirely turbulent, depending on the value of the velocity $\overline{V}$.

Lecture 2: Basic Equations of Fluid Statics and Buoyancy, Surface Tension August 27, 2025

4 Fluid Statics

4.1 The Basic Equations of Fluid Statics

Consider a differential fluid element of mass $\mathrm{d}m = \rho\,\mathrm{d}V$ with sides $\mathrm{d}x$, $\mathrm{d}y$, and $\mathrm{d}z$ as shown on Figure 4.1. The fluid element is stationary relative to the coordinate system shown.

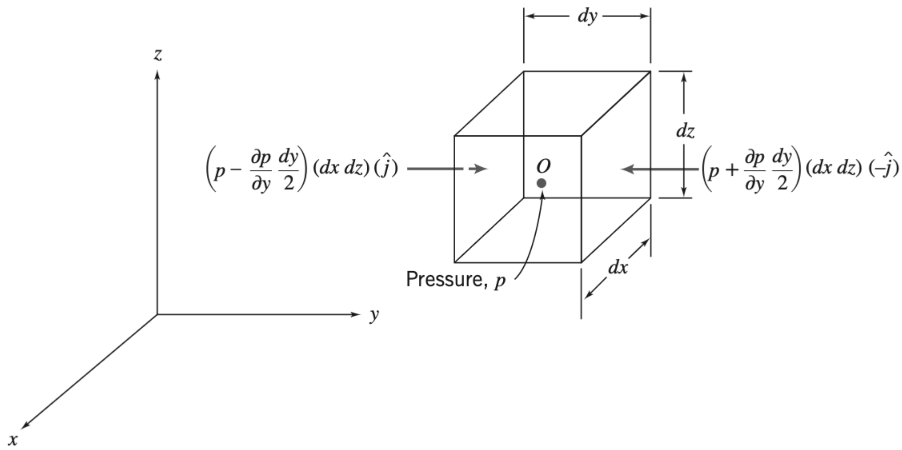


Figure 4.1: Differential fluid element with pressure forces shown in the y-direction

It has previously been mentioned that only *body forces* and *surface forces* may be applied to a fluid. In this course the body forces caused by electric or magnetic fields is assumed negligible and only the gravitational force must therefore be accounted for.

For the differential fluid element the (gravitational) body force is:

$$\mathrm{d}\mathbf{F}_B = \mathbf{g}\,\mathrm{d}m = \mathbf{g}\rho\,\mathrm{d}V.$$

where $\mathbf{g}$ denotes the local gravity vector, ρ is the density and $\mathrm{d}V$ is the volume of the fluid element. This can also be expressed as:

$$\mathrm{d}\mathbf{F}_B = \rho\mathbf{g}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$$

Per definition there can be no shear stresses in a static fluid, so the only surface force is that due to pressure, which is a scalar field, $p = p(x, y, z)$. The net pressure force can be found by summing the forces on each of the six faces of the fluid element. The pressure at the center O is denoted p. The pressure at each face of the element will be found using a Taylor series expansion of the pressure about point O. The pressure on the left side of the fluid element is:

$$p_L = p + \frac{\partial p}{\partial y}(y_L - y) = p + \frac{\partial p}{\partial y}\left(-\frac{\mathrm{d}y}{2}\right) = p - \frac{\partial p}{\partial y}\frac{\mathrm{d}y}{2}.$$

Terms of higher order are omitted as they will vanish at a later step anyways. Similarly, the pressure on the right face of the differential element is:

$$p_R = p\frac{\partial p}{\partial y}\left(y_R - y\right) = p + \frac{\partial p}{\partial y}\frac{\mathrm{d}y}{2}.$$

The pressure *forces* produced by this pressure is shown on Figure 4.1. These consist of three factors. Namely the magnitude of the pressure, the area of the face and a unit vector to indicate direction. A positive pressure here is defined as a compressive pressure on the differential element. The pressure forces in the other directions can be obtained in the same way giving:

$$\begin{aligned}\mathrm{d}\mathbf{F}_S = &\left(p - \frac{\partial p}{\partial x}\frac{\mathrm{d}x}{2}\right)(\mathrm{d}y\,\mathrm{d}z)\left(\hat{\mathbf{i}}\right) + \left(p + \frac{\partial p}{\partial x}\frac{\mathrm{d}x}{2}\right)(\mathrm{d}y\,\mathrm{d}z)\left(-\hat{\mathbf{i}}\right)\\ &+\left(p - \frac{\partial p}{\partial y}\frac{\mathrm{d}y}{2}\right)(\mathrm{d}x\,\mathrm{d}z)\left(\hat{\mathbf{j}}\right) + \left(p + \frac{\partial p}{\partial y}\frac{\mathrm{d}y}{2}\right)(\mathrm{d}x\,\mathrm{d}z)\left(-\hat{\mathbf{j}}\right)\\ &+\left(p - \frac{\partial p}{\partial z}\frac{\mathrm{d}z}{2}\right)(\mathrm{d}x\,\mathrm{d}y)\left(\hat{\mathbf{k}}\right) + \left(p + \frac{\partial p}{\partial z}\frac{\mathrm{d}z}{2}\right)(\mathrm{d}x\,\mathrm{d}y)\left(-\hat{\mathbf{k}}\right).\end{aligned}$$

Simplifying this, we obtain

$$\mathrm{d}\mathbf{F}_S = -\left(\frac{\partial p}{\partial x}\hat{\mathbf{i}} + \frac{\partial p}{\partial y}\hat{\mathbf{j}} + \frac{\partial p}{\partial z}\hat{\mathbf{k}}\right)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$$

We can now recognize the term in the parentheses as the gradient of the pressure which may be written as either $\operatorname{grad} p$ or ∇p. This is:

$$\operatorname{grad} p \equiv \nabla p \equiv \left(\hat{\mathbf{i}}\frac{\partial p}{\partial x} + \hat{\mathbf{j}}\frac{\partial p}{\partial y} + \hat{\mathbf{k}}\frac{\partial p}{\partial z}\right) \equiv \left(\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right)p.$$

Using this notation we can rewrite the expression as:

$$\mathrm{d}\mathbf{F}_S = -\operatorname{grad} p(\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z) = -\nabla p\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$$

Note that the pressure magnitude is not relevant when computing the net pressure force – instead only the rate of change of pressure with distance, the pressure gradient, is needed.

We can now combine the expressions for the surface and body forces on the fluid element.

$$\mathrm{d}\mathbf{F} = \mathrm{d}\mathbf{F}_S + \mathrm{d}\mathbf{F}_B = \left(-\nabla p + \rho\mathbf{g}\right)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \left(-\nabla p + \rho\mathbf{g}\right)\mathrm{d}V$$

which can also be stated as the force acting per unit volume as

$$\frac{\mathrm{d}\mathbf{F}}{\mathrm{d}V} = -\nabla p + \rho\mathbf{g}.$$

Applying Newton's second law to a static fluid particle gives

$$\mathbf{F} = \mathbf{a}\,\mathrm{d}m = \mathbf{0}\,\mathrm{d}m = \mathbf{0}.$$

Thus

$$\frac{\mathrm{d}\mathbf{F}}{\mathrm{d}V} = \rho\mathbf{a} = \mathbf{0}.$$

Combining these two gives:

$$-\nabla p + \rho\mathbf{g} = \mathbf{0}.$$

This can be stated in words as the sum of the net pressure force per unit volume at a point and the body force per unit volume at a point is zero. As this is a three-dimensional vector equation it is comprised of three individual equations that must each be satisfied individually.

$$-\frac{\partial p}{\partial x} + \rho g_x = 0$$
$$-\frac{\partial p}{\partial y} + \rho g_y = 0$$
$$-\frac{\partial p}{\partial z} + \rho g_z = 0.$$

It is now convenient to choose a coordinate system such that the gravity vector is aligned with one of the axes. If we place the coordinate system "normally" we would get $g_x = 0$, $g_y = 0$, and $g_z = -g$. The component equations then become

$$\frac{\partial p}{\partial x} = 0$$
$$\frac{\partial p}{\partial y} = 0$$
$$\frac{\partial p}{\partial z} = -\rho g.$$

This means that the pressure depends on one variable only and the total derivative may therefore be employed instead of the partial as

$$\frac{\partial p}{\partial z} = -\rho g \equiv -\gamma.$$

This is the basic pressure-height relation of fluid statics.

4.2 Pressure Variation in a Static Fluid

4.2.1 Incompressible Fluids

For an incompressible fluid ρ = constant. If we assume the gravity to be constant with elevation then we get

$$\frac{\partial p}{\partial z} = -\rho g = \text{constant}.$$

Now we can easily integrate this as

$$\int_{p_0}^{p} \mathrm{d}p = -\int_{z_0}^{z} \rho g \,\mathrm{d}z.$$

The height difference is defined as $h = z_0 - z$ and thus we upon integration obtain

$$p - p_0 = \Delta p = \rho g h.$$

To find the pressure difference Δp between two points separated by a series of fluid, we can compute the change in pressure as

$$\Delta p = g \sum_i \rho_i h_i.$$

4.2.2 Gases

Pressure variation in a compressible fluid must be found by integration of the previous result. First, however, an expression for the density as a function of either p or z must be found. To do this we employ the ideal gas law:

$$p = \rho R T.$$

Here the temperature T is introduced as an additional variable. In the U.S. Standard Atmosphere, the temperature decreases linearly with altitude up to an elevation of 11,0 km. Therefore the temperature can be expressed as:

$$T = T_0 - mz.$$

We therefore get

$$\mathrm{d}p = -\rho g\,\mathrm{d}z = -\frac{pg}{RT}\,\mathrm{d}z = -\frac{pg}{R(T_0 - mz)}\,\mathrm{d}z.$$

By separation of variables we obtain:

$$\int_{p_0}^{p} \frac{\mathrm{d}p}{p} = -\int_0^z \frac{g\,\mathrm{d}z}{R(T_0 - mz)}.$$

Which gives

$$\ln\frac{p}{p_0} = \frac{g}{mR}\ln\left(\frac{T_0 - mz}{T_0}\right) = \frac{g}{mR}\ln\left(1 - \frac{mz}{T_0}\right).$$

Which can be solved for p as

$$p = p_0\left(1 - \frac{mz}{T_0}\right)^{\frac{g}{mR}} = p_0\left(\frac{T}{T_0}\right)^{\frac{g}{mR}}.$$

4.3 Buoyancy

Any object submerged in a liquid will experience a vertical force acting on it due to liquid pressure – this is termed *buoyancy*. Consider the object on Figure 4.2 which is split into cylindrical volume elements.

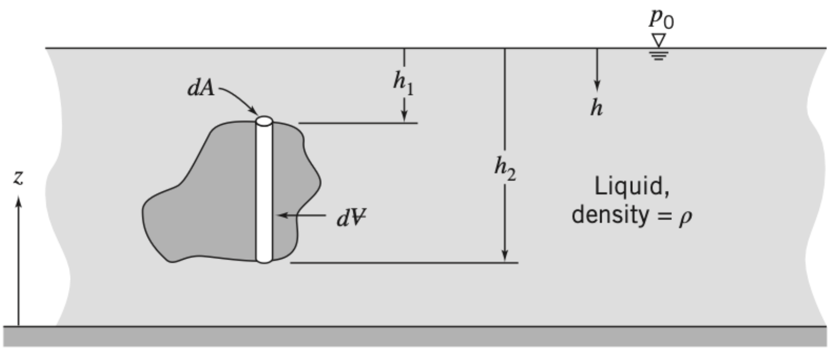


Figure 4.2: Immersed body in a static liquid.

We have previously derived an equation for the pressure p at depth h in a liquid:

$$p = p_0 + \rho g h.$$

The net vertical force on the element is thus

$$\mathrm{d}F_z = \left(p_0 + \rho g h_2\right)\mathrm{d}A - \left(p_0 + \rho g h_1\right)\mathrm{d}A = \rho g\,(h_2 - h_1)\,\mathrm{d}A.$$

As $(h_2 - h_1)\,\mathrm{d}A = \mathrm{d}V$ we get

$$F_z = \int \mathrm{d}F_z = \int_V \rho g\,\mathrm{d}V = \rho g V.$$

I.e. for a submerged body, the buoyancy force is equal to the weight of the displaced fluid.

Lecture 3: Control Volume Analysis I: Basic Laws and Mass Conservation September 1, 2025

6 Basic Equations in Integral Form for a Control Volume

In general, there are two approaches to study flowing fluids. Either one can study how an individual particle or group of particles move through space, this is often called the *system approach*. This often leds to one needing to solve a set of partial differential equations.

One can also choose to study a region of space as fluid flows through it; this is the *control volume* approach. This has widespread applications, e.g. in aerodynamics where the focus is often on the lift and drag on a wing rather than the individual fluid particles.

6.1 Basic Laws for a System

A few basic laws will be applied; these are conservation of mass, Newton's second law, the angular-momentum principle, and the first and second laws of thermodynamics. It turns out that to convert these system equations to equivalent control volume formulas we will express the properties of the system in terms of the rates of flow in and out, hence these equations are termed *rate equations*.

6.1.1 Conservation of Mass

For a system we have the simple result that $M = \text{constant}$. To express this law as a rate equation we write:

$$\left.\frac{\mathrm{d}M}{\mathrm{d}t}\right)_{\text{system}} = 0$$

where

$$M_{\text{system}} = \int_{M(\text{system})} \mathrm{d}m = \int_{V(\text{system})} \rho\,\mathrm{d}V.$$

6.1.2 Newton's Second Law

For a system (the fluid) moving relative to an inertial reference frame (the control volume), Newton's second law states that the sum of all external forces acting on the system is equal to the time rate of change of the linear momentum of the system,

$$\mathbf{F} = \left.\frac{\mathrm{d}\mathbf{P}}{\mathrm{d}t}\right)_{\text{system}}$$

where the linear momentum of the system if given by

$$\mathbf{P}_{\text{system}} = \int_{M(\text{system})} \mathbf{V}\,\mathrm{d}m = \int_{V(\text{system})} \mathbf{V}\rho\,\mathrm{d}V.$$

6.1.3 The Angular-Momentum Principle

The angular-momentum principle for a system states that the rate change of angular momentum is equal to the sum of all torques acting on the system:

$$\mathbf{T} = \left.\frac{\mathrm{d}\mathbf{H}}{\mathrm{d}t}\right)_{\text{system}}$$

where the angular momentum of the system is given by

$$\mathbf{H}_{M(\text{system})}\mathbf{r}\times\mathbf{V}\,\mathrm{d}m = \int_{V(\text{system})} \mathbf{r}\times\mathbf{V}\rho\,\mathrm{d}V.$$

Torque can be produced both by surface and body forces and also by shafts that cross the system boundary,

$$\mathbf{T} = \mathbf{r}\times\mathbf{F}_s + \int_{M(\text{system})} \mathbf{r}\times\mathbf{g}\,\mathrm{d}m + \mathbf{T}_{\text{shaft}}.$$

6.1.4 The First Law of Thermodynamics

The first law of thermodynamics is a statement of conservation of energy for a system,

$$\delta Q - \delta W = \mathrm{d}E.$$

Which in rate form is

$$\dot{Q} - \dot{W} = \left.\frac{\mathrm{d}E}{\mathrm{d}t}\right)_{\text{system}}$$

where the total energy of the system is

$$E_{\text{system}} = \int_{M(\text{system})} e \,\mathrm{d}m = \int_{V(\text{system})} e\rho \,\mathrm{d}V$$

and

$$e = u + \frac{V^2}{2} + gz.$$

Here $\dot{Q}$ is positive when heat is added to the system; $\dot{W}$ is positive when work is done by the system on its surroundings; u is the specific internal energy, V the speed, and z the height relative to a particle of substance with mass $\mathrm{d}m$.

6.1.5 The Second Law of Thermodynamics

When an amount of heat, δQ, is transferred to a system at temperature T, the second law of thermodynamics states that the change in entropy, $\mathrm{d}S$, of the system satisfies,

$$\mathrm{d}S \geq \frac{\delta Q}{T}.$$

On a rate basis this is

$$\left.\frac{\mathrm{d}S}{\mathrm{d}t}\right)_{\text{system}} \geq \frac{1}{T}\dot{Q}$$

where the total entropy of the system is given by

$$S_{\text{system}} = \int_{M(\text{system})} s \,\mathrm{d}m = \int_{V(\text{system})} s\rho \,\mathrm{d}V.$$

6.1.6 Relation of System Derivatives to the Control Volume Formulation

Let any of the parameters $M, \mathbf{P}, \mathbf{H}, E$, or S be represented by the symbol N. Corresponding to the extensive property (N) that we are trying to find we, will need the intensive (i.e., per unit mass) property η. Thus:

$$N_{\text{system}} = \int_{M(\text{system})} \eta \,\mathrm{d}m = \int_{V(\text{system})} \eta\rho \,\mathrm{d}V.$$

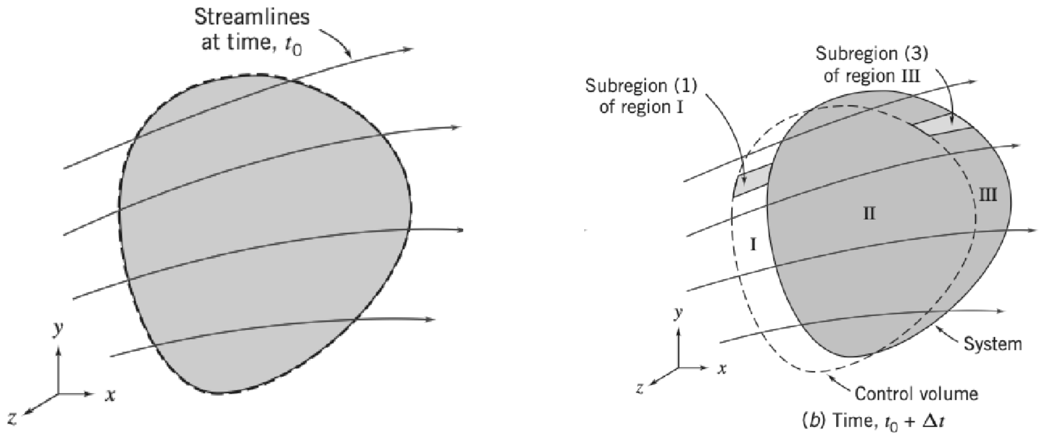
I.e. if:

$$\begin{aligned}
N &= M, & &\text{then}\,\eta = 1 \\
N &= \mathbf{P}, & &\text{then}\,\eta = \mathbf{V} \\
N &= \mathbf{H}, & &\text{then}\,\eta = \mathbf{r} \times \mathbf{V} \\
N &= E, & &\text{then}\,\eta = e \\
N &= S, & &\text{then}\,\eta = s.
\end{aligned}$$

6.1.7 Derivation

Let t represent the initial time for the system and $t + \Delta t$ represent a small time increment later. Our objective here is to relate any arbitrary extensive property, N, of the system to quantities associated with the control volume. From the definition of the derivative, the rate of change of N_{system} is

$$\left.\frac{\mathrm{d}N}{\mathrm{d}t}\right)_{\text{system}} \equiv \lim_{\Delta t \to 0} \frac{N_s)_{t_0+\Delta t} - N_s)_{t_0}}{\Delta t} \tag{8}$$



(a) Time, t_0 (b) Time, $t_0 + \Delta t$

Figure 6.1: Configuration of the system and control volume.

Here subscript s is used to denote the system.

From the geometry of Figure 6.1,

$$N_s)_{t_0+\Delta t} = (N_{\mathrm{II}} + N_{\mathrm{III}})_{t_0+\Delta t} = (N_{\mathrm{CV}} - N_{\mathrm{I}} + N_{\mathrm{III}})_{t_0+\Delta t}$$

and

$$N_s)_{t_0} = (N_{\mathrm{CV}})_{t_0}.$$

Substituting these into the definition of the system derivative in Equation (8) we get

$$\left.\frac{\mathrm{d}N}{\mathrm{d}t}\right)_s = \lim_{\Delta t\to 0} \frac{(N_{\mathrm{CV}} - N_{\mathrm{I}} + N_{\mathrm{III}})_{t_0+\Delta t} - N_{\mathrm{CV}})_{t_0}}{\Delta t}.$$

Which simplifies to

$$\left.\frac{\mathrm{d}N}{\mathrm{d}t}\right)_s = \lim_{\Delta t\to 0} \frac{N_{\mathrm{CV}})_{t_0+\Delta t} - N_{\mathrm{CV}})_{t_0}}{\Delta t} + \lim_{\Delta t\to 0} \frac{N_{\mathrm{III}})_{t_0+\Delta t}}{\Delta t} - \lim_{\Delta t\to 0} \frac{N_{\mathrm{I}})_{t_0+\Delta t}}{\Delta t}.$$

Each of the three terms can now be evaluated individually. We start with term 1, which simplifies to:

$$\lim_{\Delta t\to 0} \frac{N_{\mathrm{CV}})_{t_0+\Delta t} - N_{\mathrm{CV}})_{t_0}}{\Delta t} = \frac{\partial N_{\mathrm{CV}}}{\partial t} = \frac{\partial}{\partial} \int_{\mathrm{XC}} \eta\rho \,\mathrm{d}V.$$

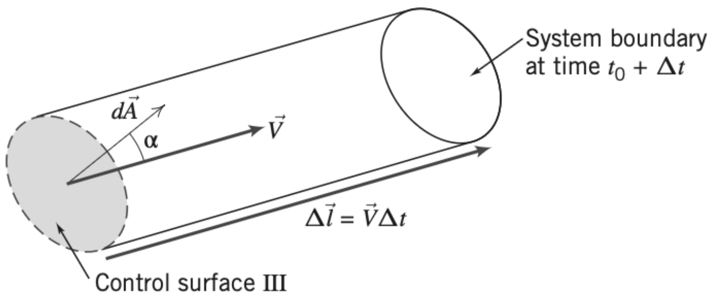


Figure 6.2: Enlarged view of subregion 3 from Figure 6.1

To evaluate term 2 we first need develop an expression for $N_{\mathrm{III}})_{t_0+\Delta t}$ by looking at the subregion 3 of region III shown on Figure 6.2. The vector area element d**A** of the control surface has magnitude dA and its direction is normal outward of the area element. The velocity vector **V** will be at some angle α with respect to d**A**

For this subregion we have

$$\mathrm{d}N_{\mathrm{III}})_{t_0+\Delta t} = \left(\eta\rho\,\mathrm{d}V\right)_{t_0+\Delta t}.$$

To obtain an expression for the volume $\mathrm{d}V$ of this cylindrical element we first note that its vector length is given by $\Delta\mathbf{l} = \mathbf{V}\Delta t$. Furthermore, the volume of a prismatic cylinder, whose area $\mathrm{d}\mathbf{A}$ is at an angle α to its length $\Delta\mathbf{l}$, is given by $\mathrm{d}V = \Delta l\,\mathrm{d}A\cos\alpha = \Delta\mathbf{l}\cdot\mathrm{d}\mathbf{A}\Delta t$.

Then for the entire region III we can integrate and obtain the second term as:

$$\lim_{\Delta t\to 0}\frac{N_{\mathrm{III}})_{t_0+\Delta t}}{\Delta t} = \lim_{\Delta t\to 0}\frac{\int_{\mathrm{CS_{III}}}\mathrm{d}N_{\mathrm{III}})_{t_0+\Delta t}}{\Delta t} = \int_{\mathrm{CS_{III}}}\eta\rho\mathbf{V}\cdot\mathrm{d}\mathbf{A}.$$

We can perform a similar analysis for subregion 1 of region I and obtain term 3 in the equation as:

$$\lim_{\Delta t\to 0}\frac{N_{\mathrm{I}})_{t_0+\Delta t}}{\Delta t} = -\int_{\mathrm{CS_I}}\eta\rho\mathbf{V}\cdot\mathrm{d}\mathbf{A}.$$

For subregion 1 the velocity vector acts into the control volume hence producing a negative scalar product. Hence the minus sign is needed to cancel the negative result of the scalar product.

We can now substitute in the three terms we have found as:

$$\left.\frac{\mathrm{d}N}{\mathrm{d}t}\right)_{\mathrm{system}} = \frac{\partial}{\partial t}\int_{\mathrm{CV}}\eta\rho\,\mathrm{d}V + \int_{\mathrm{CS_I}}\eta\rho\mathrm{V}\cdot\mathrm{d}\mathbf{A} + \int_{\mathrm{CS_{III}}}\eta\rho\mathbf{V}\cdot\mathrm{d}\mathbf{A}.$$

As $\mathrm{CS_I}$ and $\mathrm{CS_{III}}$ constitute the entire control surface the last two integrals can be combined as:

$$\left.\frac{\mathrm{d}N}{\mathrm{d}t}\right)_{\mathrm{system}} = \frac{\partial}{\partial t}\int_{\mathrm{CV}}\eta\rho\,\mathrm{d}V + \int_{\mathrm{CS}}\eta\rho\mathbf{V}\cdot\mathrm{d}\mathbf{A} \tag{9}$$

Equation (9) is the fundamental relation between the rate of change of any extensive property N of a system and the variations of this property associated with a control volume. This is also called the *Reynolds Transport Theorem*.

Here it is important to note that the system is defined as the matter that happens to be passing through the chosen control volume at the instant we chose. I.e.

- $\left.\frac{\mathrm{d}N}{\mathrm{d}t}\right)_{\mathrm{system}}$ is the rate of change of the system extensive property N, e.g. if $N = \mathbf{P}$ we obtain the rate of change of momentum
- $\frac{\partial}{\partial t}\int_{\mathrm{CV}}\eta\rho\,\mathrm{d}V$ is the rate of change of N in the control volume.
- $\int_{\mathrm{CS}}\eta\rho\mathbf{V}\cdot\mathrm{d}\mathbf{A}$ is the rate at which N is exiting the surface of the control volume.

6.2 Conservation of Mass

We remember from Section 6.1.1 that the mass of the system remains constant,

$$\left.\frac{\mathrm{d}M}{\mathrm{d}t}\right)_{\mathrm{system}} = 0.$$

If we set $N = M$ and $\eta = 1$ and plug it into Equation (9) we obtain

$$\left.\frac{\mathrm{d}M}{\mathrm{d}t}\right)_{\mathrm{system}} = \frac{\partial}{\partial t}\int_{\mathrm{CV}}\rho\,\mathrm{d}V + \int_{\mathrm{CS}}\rho\mathbf{V}\cdot\mathrm{d}\mathbf{A}.$$

And combining these two gives the control volume formulation of conservation of mass:

$$\frac{\partial}{\partial t}\int_{\mathrm{CV}}\rho\,\mathrm{d}V+\int_{\mathrm{CS}}\rho\mathbf{V}\cdot\mathrm{d}\mathbf{A}=0 \tag{10}$$

6.2.1 Special Cases

In some special cases Equation (10) can be simplified. Consider first the case of an incompressible fluid, i.e. ρ is constant in space and time. Consequently Equation (10) can be written as:

$$\rho\frac{\partial}{\partial t}\int_{\mathrm{CV}}\mathrm{d}V+\rho\int_{\mathrm{CS}}\mathbf{V}\cdot\mathrm{d}A=0.$$

The integral of $\mathrm{d}V$ over the control volume is simply the volume, this

$$\frac{\partial V}{\partial t}+\int_{\mathrm{CS}}\mathbf{V}\cdot\mathrm{d}\mathbf{A}=0.$$

For a control volume of fixed size and shape, $V=$ constant. The conservation of mass for incompressible flow through a fixed volume becomes:

$$\int_{\mathrm{CS}}\mathbf{V}\cdot\mathrm{d}\mathbf{A}=0.$$

A useful special case is when we have uniform velocity at each inlet and exit. In this case it simplifies to

$$\sum_{\mathrm{CS}}\mathbf{V}\cdot\mathbf{A}=0.$$

We now consider the case of *steady, compressible flow* through a fixed control volume. Since the flow is steady no fluid property varies with time. Consequently the first term of Equation (10) must be zero and hence for steady flow the statement of conservation of mass becomes:

$$\int_{\mathrm{CS}}\rho\mathbf{V}\cdot\mathrm{d}\mathbf{A}=0.$$

When we have uniform velocity at each inlet and exit this simplifies to:

$$\sum_{\mathrm{CS}}\rho\mathbf{V}\cdot\mathbf{A}=0.$$

Lecture 4: Control Volume Analysis II: Momentum & Energy Equation September 3, 2025

7.1 Momentum Equation for Inertial Control Volume

We will now find a control volume form of Newton's second law. Note that in the following the coordinates (with respect to which velocities are measured) are inertial, i.e. either at rest or moving at a constant speed with respect to an "absolute" set of coordinates.

We have previously defined Newton's second law for a system moving relative to an inertial coordinate system in Section 6.1.2 as:

$$\mathbf{F}=\left.\frac{\mathrm{d}\mathbf{P}}{\mathrm{d}t}\right)_{\text{system}}$$

where the linear momentum of the system is given by

$$\mathbf{P}_{\text{system}}=\int_{M(\text{system})}\mathbf{V}\,\mathrm{d}m=\int_{V(\text{system})}\mathbf{V}\rho\,\mathrm{d}V$$

and the resultant force **F** includes all surface and body forces acting on the system

$$\mathbf{F} = \mathbf{F}_S + \mathbf{F}_B.$$

We have previously derived the relation between the system and control volume formulations in Equation (9) as:

$$\left.\frac{\mathrm{d}N}{\mathrm{d}t}\right)_{\text{system}} = \frac{\partial}{\partial t}\int_{\text{CV}} \eta\rho \,\mathrm{d}V + \int_{\text{CS}} \eta\rho\mathbf{V}\cdot\mathrm{d}\mathbf{A}.$$

In this we now set $N = \mathbf{P}$ and $\eta = \mathbf{V}$. We thus obtain:

$$\left.\frac{\mathrm{d}\mathbf{P}}{\mathrm{d}t}\right)_{\text{system}} = \frac{\partial}{\partial t}\int_{\text{CV}} \mathbf{V}\rho \,\mathrm{d}V + \int_{\text{CS}} \mathbf{V}\rho\mathbf{V}\cdot\mathrm{d}\mathbf{A}.$$

Since in deriving Equation (9) the system and control volume coincided t t_0, then

$$\mathbf{F})_{\text{on system}} = \mathbf{F})_{\text{on control volume}} = \mathbf{F}_S + \mathbf{F}_B.$$

Now these can be combined as:

$$\mathbf{F} = \mathbf{F}_S + \mathbf{F}_B = \frac{\partial}{\partial t}\int_{\text{CV}} \mathbf{V}\rho \,\mathrm{d}V + \int_{\text{CS}} \mathbf{V}\rho\mathbf{V}\cdot\mathrm{d}\mathbf{A} \tag{11}$$

For cases with uniform flow at each inlet and exit this becomes:

$$\mathbf{F} = \mathbf{F}_S + \mathbf{F}_B = \frac{\partial}{\partial t}\int_{\text{CV}} \mathbf{V}\rho \,\mathrm{d}V + \sum_{\text{CS}} \mathbf{V}\rho\mathbf{V}\cdot\mathbf{A}.$$

When applying Equation (11) we need to be a little careful. First we must choose a control volume and its control surface such that the volume integral and surface integral can be evaluated. In fluid mechanics the body force is usually gravity, so

$$\mathbf{F}_B = \int_{\text{CV}} \rho\mathbf{g}\,\mathrm{d}V = \mathbf{W}_{\text{CW}} = M\mathbf{g}.$$

In many applications the surface force is due to pressure,

$$\mathbf{F}_S = \int_A -p\,\mathrm{d}\mathbf{A}.$$

The momentum equation in Equation (11) is a vector equation therefore it can be written in three scalar components as:

$$F_x = F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{\text{CV}} u\rho\,\mathrm{d}V + \int_{\text{CS}} u\rho\mathbf{V}\cdot\mathrm{d}\mathbf{A}$$
$$F_y = F_{S_y} + F_{B_y} = \frac{\partial}{\partial t}\int_{\text{CV}} v\rho\,\mathrm{d}V + \int_{\text{CS}} v\rho\mathbf{V}\cdot\mathrm{d}\mathbf{A}$$
$$F_z = F_{S_z} + F_{B_z} = \frac{\partial}{\partial t}\int_{\text{CV}} w\rho\,\mathrm{d}V + \int_{\text{CS}} w\rho\mathbf{V}\cdot\mathrm{d}\mathbf{A}.$$

Or in the case of uniform flow at each inlet and exit as

$$F_x = F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{\text{CV}} u\rho\,\mathrm{d}V + \sum_{\text{CS}} u\rho\mathbf{V}\cdot\mathbf{A}$$
$$F_y = F_{S_y} + F_{B_y} = \frac{\partial}{\partial t}\int_{\text{CV}} v\rho\,\mathrm{d}V + \sum_{\text{CS}} v\rho\mathbf{V}\cdot\mathbf{A}$$
$$F_z = F_{S_z} + F_{B_z} = \frac{\partial}{\partial t}\int_{\text{CV}} w\rho\,\mathrm{d}V + \sum_{\text{CS}} w\rho\mathbf{V}\cdot\mathbf{A}.$$

7.1.1 Differential Control Volume Analysis

The control volume approach that has been presented in the above is useful when applied to a finite region.

If we instead apply the approach to a differential control volume, we can obtain differential equations describing a flow field. Let us apply the continuity and momentum equations to a steady incompressible flow without friction as on Figure 7.1. The control volume is fixed in space and bounded by flow streamlines, and is thus an element of a stream tube. The length of the control volume is $\mathrm{d}s$.

As the control volume is bounded by streamlines, flow across the boundaries of the control volume only happens at the end sections located at coordinates s and $s+\mathrm{d}s$. Properties at the inlet sections are assigned arbitrary symbolic values and the properties at the outlet section are assumed to increase by differential amounts.

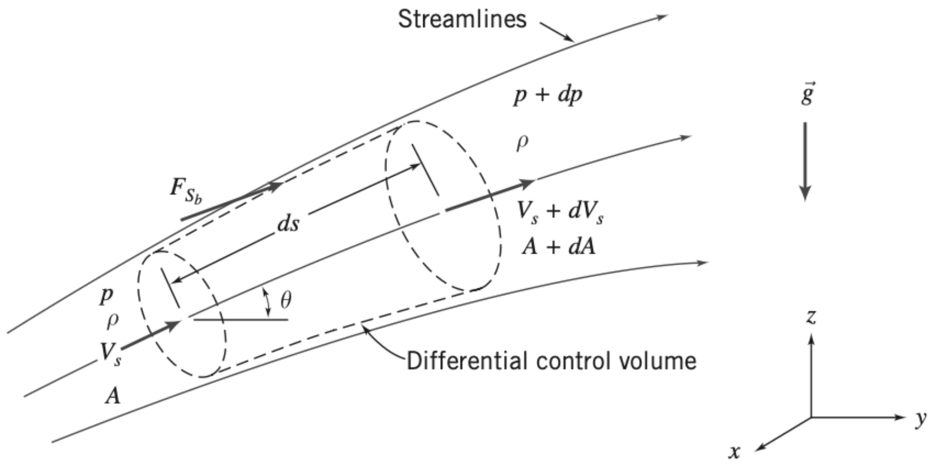


Figure 7.1: Differential control volume through a stream tube.

Now we can apply the continuity equation,

$$\frac{\partial}{\partial t}\int_{\text{CV}} \rho\,\mathrm{d}V + \int_{\text{CS}} \rho\mathbf{V}\cdot\mathrm{d}\mathbf{A} = 0.$$

Which under assumptions of steady flow, no flow across the bounding streamlines and that the flow is incompressible ($\rho = \text{constant}$), reduces to:

$$\left(-\rho V_s A\right) + \left(\rho\,(V_s + \mathrm{d}V_s)\,(A + \mathrm{d}A)\right) = 0 \implies \rho\,(V_s + \mathrm{d}V_s)\,(A + \mathrm{d}A) = \rho V_s A.$$

Simplifying we obtain

$$V_s\,\mathrm{d}A + A\,\mathrm{d}V_s + \mathrm{d}A\,\mathrm{d}V_s = 0.$$

The product of the differentials $\mathrm{d}A\,\mathrm{d}V_s \approx 0$ compared to the other terms so this can be neglected, leaving us with:

$$V_s\,\mathrm{d}A + A\,\mathrm{d}V_s = 0.$$

Using the streamwise component of the momentum equation we start with:

$$F_{S_s} + F_{B_s} = \frac{\partial}{\partial t}\int_{\text{CV}} u_s \rho \,\mathrm{d}V + \int_{\text{CS}} u_s \rho \mathbf{V}\cdot \mathrm{d}\mathbf{A}.$$

As we assume no friction F_{S_s} is due to pressure forces only, and will thus have three terms:

$$F_{S_s} = pA - \left(p + \mathrm{d}p\right)(A + \mathrm{d}A) + \left(p + \frac{\mathrm{d}p}{2}\right)\mathrm{d}A.$$

The first and second terms in this are the pressure forces on the end faces of the control surface. The third term is F_{s_b}, the pressure force acting in the s direction on the bounding stream surface. The above equation simplifies to

$$F_{S_s} = -A\,\mathrm{d}p - \frac{1}{2}\mathrm{d}p\,\mathrm{d}A.$$

The body force component in the s direction is

$$F_{B_s} = \rho g_s \,\mathrm{d}V = \rho\left(-g\sin\theta\right)\left(A + \frac{\mathrm{d}A}{2}\right)\mathrm{d}s.$$

But $\sin\theta\,\mathrm{d}s = \mathrm{d}z$ so

$$F_{B_s} = -\rho g\left(A + \frac{\mathrm{d}A}{2}\right)\mathrm{d}z.$$

The momentum flux will be

$$\int_{\text{CS}} u_s \rho \mathbf{V}\cdot \mathrm{d}\mathbf{A} = V_s\left(-\rho V_s A\right) + (V_s + \mathrm{d}V_s)\left(\rho\,(V_s + \mathrm{d}V_s)\,(A + \mathrm{d}A)\right).$$

The two mass flux factors are equal from continuity so

$$\int_{\text{CS}} u_s \rho \mathbf{V}\cdot \mathrm{d}\mathbf{A} = V_s\left(-\rho V_s A\right) + (V_s + \mathrm{d}V_s)\left(\rho V_s A\right) = \rho V_s A \mathrm{d}V_s.$$

Substituting all of these into the momentum equation yields

$$-A\,\mathrm{d}p - \frac{1}{2}\mathrm{d}p\,\mathrm{d}A - \rho g A\,\mathrm{d}z - \frac{1}{2}\rho g\,\mathrm{d}A\,\mathrm{d}z = \rho V_s A \mathrm{d}V_s.$$

Dividing by ρA and noting that products of differentials are negligible we obtain:

$$-\frac{\mathrm{d}p}{\rho} - g\,\mathrm{d}z = V_s\,\mathrm{d}V_s = d\left(\frac{V_s^2}{2}\right)$$

or

$$\frac{\mathrm{d}p}{\rho} + d\left(\frac{V_s^2}{2}\right) + g\,\mathrm{d}z = 0.$$

As the flow is incompressible and ρ = constant we can integrate this to obtain:

$$\frac{p}{\rho} + \frac{V_s^2}{2} + gz = \text{constant}.$$

This is one form of the Bernoulli equation and it is only valid under the following restrictions:

1. Steady flow
2. No friction
3. Flow along a streamline
4. Incompressible flow

7.2 Momentum equation for Control Volume with Rectilinear Acceleration

To develop the momentum equation for a linearly accelerating control volume, it is necessary to relate $\mathbf{P}_{XYZ}$ to $\mathbf{P}_{xyz}$. We begin by writing Newton's second law for a system, remembering the acceleration must be measured relative to an inertial reference frame that we have designated XYZ. We get

$$\mathbf{F} = \frac{\mathrm{d}\mathbf{P}_{XYZ}}{\mathrm{d}t}\bigg)_{\text{system}} = \frac{\mathrm{d}}{\mathrm{d}t}\int_{M(\text{system})} \mathbf{V}_{XYZ}\,\mathrm{d}m = \int_{M(\text{system})} \frac{\mathrm{d}\mathbf{V}_{XYZ}}{\mathrm{d}t}\,\mathrm{d}m.$$

The velocities with respect to the inertial (XYZ) and the control volume coordinates (xyz) are related by the relative-motion equation:

$$\mathbf{V}_{XYZ} = \mathbf{V}_{xyz} + \mathbf{V}_{rf}$$

where $\mathbf{V}_{rf}$ is the velocity of the control volume coordinates xyz with respect to the absolute XYZ.

We assume the motion of xyz is purely translational relative to the inertial reference frame XYZ, so:

$$\frac{\mathrm{d}\mathbf{V}_{XYZ}}{\mathrm{d}t} = \mathbf{a}_{XYZ} = \frac{\mathrm{d}\mathbf{V}_{xyz}}{\mathrm{d}t} + \frac{\mathrm{d}\mathbf{V}_{rf}}{\mathrm{d}t} = \mathbf{a}_{xyz} + \mathbf{a}_{rf}.$$

Substituting this into Newton's second law from above we get:

$$\mathbf{F} = \int_{M(\text{system})} \mathbf{a}_{rf}\,\mathrm{d}m + \int_{M(\text{system})} \frac{\mathrm{d}\mathbf{V}_{xyz}}{\mathrm{d}t}\,\mathrm{d}m$$

or

$$\mathbf{F} - \int_{M(\text{system})} \mathbf{a}_{rf}\,\mathrm{d}m = \frac{\mathrm{d}\mathbf{P}_{xyz}}{\mathrm{d}t}\bigg)_{\text{system}}$$

where the linear momentum of the system is

$$\mathbf{P}_{xyz})_{\text{system}} = \int_{M(\text{system})} \mathbf{V}_{xyz}\,\mathrm{d}m = \int_{V(\text{system})} \mathbf{V}_{xyz}\rho\,\mathrm{d}V$$

and the force $\mathbf{F}$ includes all surface and body forces that are acting on the system.

To derive the control volume formulation of Newton's second law, we set $N = \mathbf{P}_{xyz}$ and $\eta = \mathbf{V}_{xyz}$. Substituting this into Equation (9) we obtain:

$$\frac{\mathrm{d}\mathbf{P}_{xyz}}{\mathrm{d}t}\bigg)_{\text{system}} = \frac{\partial}{\partial t}\int_{\text{CV}} \mathbf{V}_{xyz}\rho\,\mathrm{d}V + \int_{\text{CS}} \mathbf{V}_{xyz}\rho\mathbf{V}_{xyz}\cdot\mathrm{d}\mathbf{A}.$$

If we combine this with the linear momentum equation for the system we obtain:

$$\mathbf{F} - \int_{\text{CV}} \mathbf{a}_{rf}\rho\,\mathrm{d}V = \frac{\partial}{\partial t}\int_{\text{CV}} \mathbf{V}_{xyz}\rho\,\mathrm{d}V + \int_{\text{CS}} \mathbf{V}_{xyz}\rho\mathbf{V}_{xyz}\cdot\mathrm{d}\mathbf{A}.$$

And since $\mathbf{F} = \mathbf{F}_S + \mathbf{F}_B$ this becomes

$$\mathbf{F}_S + \mathbf{F}_B - \int_{\text{CV}} \mathbf{a}_{rf}\rho\,\mathrm{d}V = \frac{\partial}{\partial t}\int_{\text{CV}} \mathbf{V}_{xyz}\rho\,\mathrm{d}V + \int_{\text{CS}} \mathbf{V}_{xyz}\rho\mathbf{V}_{xyz}\cdot\mathrm{d}\mathbf{A} \tag{12}$$

Comparing Equation (12) to that for a non-accelerating control volume we see that they only differ by the introduction of the term $-\int_{\text{CV}} \mathbf{a}_{rf}\rho\,\mathrm{d}V$, and for a non-accelerating reference frame $\mathbf{a}_{rf} = 0$ and it reduces to the equation for a non-accelerating reference frame.

This can also be written in components as:

$$
\begin{aligned}
F_{S_x} + F_{B_x} - \int_{\text{CV}} a_{rf_x}\rho\,\mathrm{d}V &= \frac{\partial}{\partial t}\int_{\text{CV}} u_{xyz}\rho\,\mathrm{d}V + \int_{\text{CS}} u_{xyz}\rho\mathbf{V}_{xyz}\cdot\mathrm{d}\mathbf{A} \\
F_{S_y} + F_{B_y} - \int_{\text{CV}} a_{rf_y}\rho\,\mathrm{d}V &= \frac{\partial}{\partial t}\int_{\text{CV}} v_{xyz}\rho\,\mathrm{d}V + \int_{\text{CS}} v_{xyz}\rho\mathbf{V}_{xyz}\cdot\mathrm{d}\mathbf{A} \\
F_{S_z} + F_{B_z} - \int_{\text{CV}} a_{rf_z}\rho\,\mathrm{d}V &= \frac{\partial}{\partial t}\int_{\text{CV}} w_{xyz}\rho\,\mathrm{d}V + \int_{\text{CS}} w_{xyz}\rho\mathbf{V}_{xyz}\cdot\mathrm{d}\mathbf{A}.
\end{aligned}
$$

7.3 The Angular-Momentum Principle

7.3.1 Equation for Fixed Control Volume

The angular-momentum principle for a system in an inertial frame is

$$\mathbf{T} = \left.\frac{\mathrm{d}\mathbf{H}}{\mathrm{d}t}\right)_{\text{system}}$$

where $\mathbf{T}$ is the total torque exerted on the system by its surroundings and $\mathbf{H}$ is the angular momentum of the system.

$$\mathbf{H} = \int_{M(\text{system})} \mathbf{r}\times\mathbf{V}\,\mathrm{d}m = \int_{V(\text{system})} \mathbf{r}\times\mathbf{V}\rho\,\mathrm{d}V.$$

If we let $\mathbf{r}$ locate each mass or volume element of the system with respect to the coordinate system, then the torque $\mathbf{T}$ applied to the system may be written:

$$\mathbf{T} = \mathbf{r}\times\mathbf{F}_s + \int_{M(\text{system})} \mathbf{r}\times\mathbf{g}\,\mathrm{d}m + \mathbf{T}_{\text{shaft}}.$$

The relation between the system and fixed control volume formulation is

$$\left.\frac{\mathrm{d}N}{\mathrm{d}t}\right)_{\text{system}} = \frac{\partial}{\partial t}\int_{\text{CV}} \eta\rho\,\mathrm{d}V + \int_{\text{CS}} \eta\rho\mathbf{V}\cdot\mathrm{d}\mathbf{A}.$$

where $N_{\text{system}} = \int_{M(\text{system})} \eta\,\mathrm{d}m$. If we set $N = \mathbf{H}$ and $\eta = \mathbf{r}\times\mathbf{V}$ then:

$$\left.\frac{\mathrm{d}\mathbf{H}}{\mathrm{d}t}\right)_{\text{system}} = \frac{\partial}{\partial t}\int_{\text{CV}} \mathbf{r}\times\mathbf{V}\rho\,\mathrm{d}V + \int_{\text{CS}} \mathbf{r}\times\mathbf{V}\rho\mathbf{V}\cdot\mathrm{d}\mathbf{A}.$$

Combining these we obtain:

$$\mathbf{r}\times\mathbf{F}_s + \int_{M(\text{system})} \mathbf{r}\times\mathbf{g}\,\mathrm{d}m + \mathbf{T}_{\text{shaft}} = \frac{\partial}{\partial t}\int_{\text{CV}} \mathbf{r}\times\mathbf{V}\rho\,\mathrm{d}V + \int_{\text{CS}} \mathbf{r}\times\mathbf{V}\rho\mathbf{V}\cdot\mathrm{d}\mathbf{A}.$$

Since the system and control volume coincide at t_0 we have that $\mathbf{T} = \mathbf{T}_{\text{CV}}$ and therefore:

$$\mathbf{r}\times\mathbf{F}_s + \int_{\text{CV}} \mathbf{r}\times\mathbf{g}\rho\,\mathrm{d}V + \mathbf{T}_{\text{shaft}} = \frac{\partial}{\partial t}\int_{\text{CV}} \mathbf{r}\times\mathbf{V}\rho\,\mathrm{d}V + \int_{\text{CS}} \mathbf{r}\times\mathbf{V}\rho\mathbf{V}\cdot\mathrm{d}\mathbf{A}.$$

7.4 The First Law of Thermodynamics

We recall the system formulation of the first law of thermodynamics was

$$\dot{Q} - \dot{W} = \left.\frac{\mathrm{d}E}{\mathrm{d}t}\right)_{\text{system}}$$

where the total energy of the system is given by

$$E_{\text{system}} = \int_{M(\text{system})} e\,\mathrm{d}m = \int_{V(\text{system})} e\rho\,\mathrm{d}V$$

and

$$e = u + \frac{V^2}{2} + gz.$$

We set $N = E$ and $\eta = e$ in Equation (9) and obtain:

$$\left.\frac{\mathrm{d}E}{\mathrm{d}t}\right)_{\text{system}} = \frac{\partial}{\partial t}\int_{\text{CV}} e\rho\,\mathrm{d}V + \int_{\text{CS}} e\rho\mathbf{V}\cdot\mathrm{d}\mathbf{A}.$$

Since the system and control volume coincide at t_0 we have that $[\dot{Q} - \dot{W}]_{\text{system}} = [\dot{Q} - \dot{W}]_{\text{control volume}}$. In light of this we get the control volume form of the first law of thermodynamics as:

$$\dot{Q} - \dot{W} = \frac{\partial}{\partial t}\int_{\text{CV}} e\rho\,\mathrm{d}V + \int_{\text{CS}} e\rho\mathbf{V}\cdot\mathrm{d}\mathbf{A} \tag{13}$$

where

$$e = u + \frac{V^2}{2} + gz.$$

Note that for steady flow the right hand side of Equation (13) is zero.

7.4.1 Rate of Work Done by a Control Volume

The rate of work done by a control volume is subdivided into four classifications,

$$\dot{W} = \dot{W}_s + \dot{W}_{\text{normal}} + \dot{W}_{\text{shear}} + \dot{W}_{\text{other}}.$$

Shaft Work We will designate the shaft work W_s and hence the rate of work transferred out through the control system by shaft work is designated $\dot{W}_s$.

Work Done by Normal Stresses at the Control Surface Work requires a force to act through a distance. Thus, when a force $\mathbf{F}$ acts through an infinitesimal displacement d$\mathbf{s}$ the work done is given by

$$\delta W = \mathbf{F}\cdot\mathrm{d}\mathbf{s}.$$

If we divide this by the time increment Δt and take the limit as $\Delta t \to 0$ we obtain the rate of work done by the force $\mathbf{F}$ as:

$$\dot{W} = \lim_{\Delta t\to 0}\frac{\delta W}{\Delta T} = \lim_{\Delta t\to 0}\frac{\mathbf{F}\cdot\mathrm{d}\mathbf{s}}{\Delta t} \implies \dot{W} = \mathbf{F}\cdot\mathbf{V}.$$

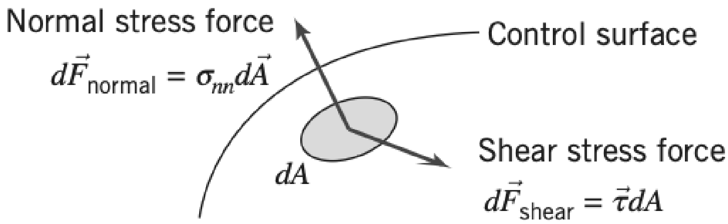


Figure 7.2: Normal and shear stress forces.

We can use this to compute the rate of work done by normal and shear stresses. We consider the segment og control surface shown on Figure 7.2. For an element of area d**A** we can write an expression for the normal stress force $\mathrm{d}\mathbf{F}_{\text{normal}}$. This will be given by the normal stress σ_{nn} multiplied by the vector element d**A**. Hence:

$$\mathrm{d}\mathbf{F}_{\text{normal}} \cdot \mathbf{V} = \sigma_{nn}\,\mathrm{d}\mathbf{A} \cdot \mathbf{V}.$$

Since the work out from the control volume is the negative work done on the control volume we get:

$$\dot{W}_{\text{normal}} = -\int_{\text{CS}} \sigma_{nn}\,\mathrm{d}\mathbf{A} \cdot \mathbf{V} = -\int_{\text{CS}} \sigma_{nn}\mathbf{V} \cdot \mathrm{d}\mathbf{A}.$$

Work Done by Shear Stresses at the Control Surface As shown on Figure 7.2 the shear force acting on an element of the control surface is given by:

$$\mathrm{d}\mathbf{F}_{\text{shear}} = \boldsymbol{\tau}\,\mathrm{d}A$$

where the shear stress vector, $\boldsymbol{\tau}$, is the shear stress acting in some direction in the plane of $\mathrm{d}A$. The rate of work done on the entire control surface by shear stresses is thus:

$$\int_{\text{CS}} \boldsymbol{\tau}\,\mathrm{d}A \cdot \mathbf{V} = \int_{\text{CS}} \boldsymbol{\tau} \cdot \mathbf{V}\,\mathrm{d}A.$$

Since the work out from the control volume is the negative of the work done on the control volume we get:

$$\dot{W}_{\text{shear}} = -\int_{\text{CS}} \boldsymbol{\tau} \cdot \mathbf{V}\,\mathrm{d}A.$$

This is better expressed as three terms:

$$\dot{W}_{\text{shear}} = -\int_{\text{CS}} \boldsymbol{\tau} \cdot \mathbf{V}\,\mathrm{d}A = -\int_{A(\text{shafts})} \boldsymbol{\tau} \cdot \mathbf{V}\,\mathrm{d}A - \int_{A(\text{solid surface})} \boldsymbol{\tau} \cdot \mathbf{V}\,\mathrm{d}A - \int_{A(\text{ports})} \boldsymbol{\tau} \cdot \mathbf{V}\,\mathrm{d}A.$$

We have already accounted for the first term as $\dot{W}_s$ previously. At solid surfaces, $\mathbf{V} = 0$, so the second term is zero (for a fixed control volume). Thus

$$\dot{W}_{\text{shear}} = -\int_{A(\text{ports})} \boldsymbol{\tau} \cdot \mathbf{V}\,\mathrm{d}A.$$

For a control surface perpendicular to **V** we get $\boldsymbol{\tau} \cdot \mathbf{V} = 0$ and therefore $\dot{W}_{\text{shear}} = 0$.

Other Work This includes electrical and electromagnetic energy that could be absorbed by the control volume. This is absent in most cases, but for the general formulation it must be included.

With all terms in $\dot{W}$ evaluated, we get:

$$\dot{W} = \dot{W}_s - \int_{\text{CS}} \sigma_{nn}\mathbf{V} \cdot \mathrm{d}\mathbf{A} + \dot{W}_{\text{shear}} + \dot{W}_{\text{other}}.$$

Substituting this into the original expression we get:

$$\dot{Q} - \dot{W}_s + \int_{\text{CS}} \sigma_{nn}\mathbf{V} \cdot \mathrm{d}\mathbf{A} - \dot{W}_{\text{shear}} - \dot{W}_{other} = \frac{\partial}{\partial t}\int_{\text{CV}} e\rho\,\mathrm{d}V + \int_{\text{CS}} e\rho\mathbf{V} \cdot \mathrm{d}\mathbf{A}.$$

Rearranging this and remembering $\rho = \frac{1}{v}$, where v is specific volume we get:

$$\dot{Q} - \dot{W}_s - \dot{W}_{\text{shear}} - \dot{W}_{\text{other}} = \frac{\partial}{\partial t}\int_{\text{CV}} e\rho\,\mathrm{d}V + \int_{\text{CS}} \left(e + \rho v\right)\rho\mathbf{V} \cdot \mathrm{d}\mathbf{A}.$$

Substituting $e = u + \frac{V^2}{2} + gz$ into the last term we obtain the first law of thermodynamics for a control volume as:

$$\dot{Q} - \dot{W}_s - \dot{W}_{\text{shear}} - \dot{W}_{\text{other}} = \frac{\partial}{\partial t}\int_{\text{CV}} e\rho \,\mathrm{d}V + \int_{\text{CS}} \left(u + pv + \frac{V^2}{2} + gz\right)\rho \mathbf{V}\cdot \mathrm{d}\mathbf{A}.$$

7.4.2 The Second Law of Thermodynamics

The second law of thermodynamics applies to all fluid systems and is formulated as:

$$\left.\frac{\mathrm{d}S}{\mathrm{d}t}\right)_{\text{system}} \geq \frac{1}{T}\dot{Q}$$

where the total entropy of the system is given by

$$S_{\text{system}} = \int_{M(\text{system})} s \,\mathrm{d}m = \int_{V(\text{system})} s\rho \,\mathrm{d}V.$$

We set $N = S$ and $\eta = s$ in Equation (9) and obtain

$$\left.\frac{\mathrm{d}S}{\mathrm{d}t}\right)_{\text{system}} = \frac{\partial}{\partial t}\int_{\text{CV}} s\rho \,\mathrm{d}V + \int_{\text{CS}} s\rho \mathbf{V}\cdot \mathrm{d}\mathbf{A}.$$

As the system and control volume coincide at t_0 we have that

$$\frac{1}{T}\dot{Q})_{\text{system}} = \frac{1}{T}\dot{Q})_{\text{CV}} = \int_{\text{CS}} \frac{1}{T}\left(\frac{\dot{Q}}{A}\right) \mathrm{d}A.$$

Thus the control volume formulation of the second law of thermodynamics is:

$$\frac{\partial}{\partial t}\int_{\text{CV}} s\rho \,\mathrm{d}V + \int_{\text{CS}} s\rho \mathbf{V}\cdot \mathrm{d}\mathbf{A} \geq \int_{\text{CS}} \frac{1}{T}\left(\frac{\dot{Q}}{A}\right) \mathrm{d}A.$$

Here the factor $\frac{\dot{Q}}{A}$ represents the heat flux per unit area into the control volume through the area element $\mathrm{d}A$. To evaluate the term

$$\int_{\text{CS}} \frac{1}{T}\left(\frac{\dot{Q}}{A}\right) \mathrm{d}A.$$

both the local heat flux and local temperature T must be known for each area element of the control surface.

Lecture 5: Differential Analysis I: Mass Conservation September 8, 2025

9 Introduction to Differential Analysis of Fluid Motion

In the previous Section 6 the basic equations in integral form for a control volume were introduced. These are useful when we are interested in the overall behaviour of a fluid field and its effects on its surroundings. To determine what happens at each point of the flow field we must ude the differential forms of the equations of motion.

9.1 Conservation of Mass

9.1.1 Rectangular Coordinate System

To find the differential equation for conservation of mass in a rectangular coordinate system we choose a control volume that is an infinitesimal cube with sides of length $\mathrm{d}x, \mathrm{d}y, \mathrm{d}z$ as shown on Figure 9.1. The density at the center, O, of the control volume is assumed to be ρ and the velocity there is assumed to be $\mathbf{V} = \hat{\imath}u + \hat{\jmath}v + \hat{k}w$.

Now, to evaluate the properties at each of the six faces of the control series we will use a Taylor series expansion

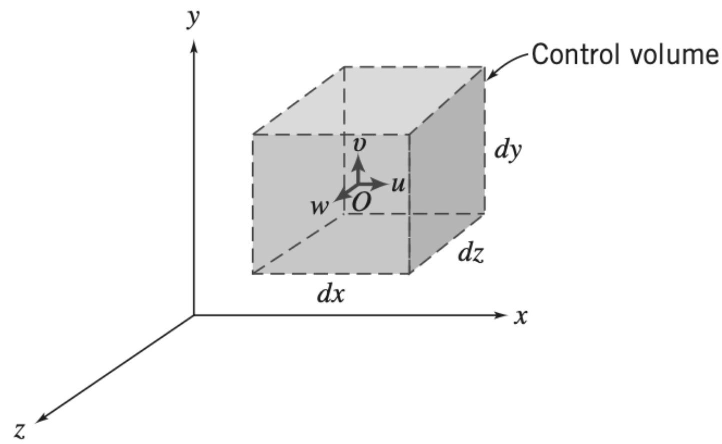


Figure 9.1: Differential control volume in rectangular coordinates.

about point O. E.g. for the right face:

$$\rho)_{x+\frac{\mathrm{d}x}{2}} = \rho + \left(\frac{\partial \rho}{\partial x}\right)\frac{\mathrm{d}x}{2} + \left(\frac{\partial^2 \rho}{\partial x^2}\right)\frac{1}{2!}\left(\frac{\mathrm{d}x}{2}\right)^2 + \ldots.$$

By neglecting higher-order terms, we can write:

$$\rho)_{x+\frac{\mathrm{d}x}{2}} = \rho + \left(\frac{\partial \rho}{\partial x}\right)\frac{\mathrm{d}x}{2}.$$

Likewise for the velocity we can write:

$$u)_{x+\frac{\mathrm{d}x}{2}} = u + \left(\frac{\partial u}{\partial x}\right)\frac{\mathrm{d}x}{2}$$

where ρ, u, $\frac{\partial p}{\partial x}$, and $\frac{\partial u}{\partial x}$ are all evaluated at point O. Following the same procedure for the left hand side we get:

$$\rho)_{x-\frac{\mathrm{d}x}{2}} = \rho - \left(\frac{\partial p}{\partial x}\right)\frac{\mathrm{d}x}{2}$$
$$u)_{x-\frac{\mathrm{d}x}{2}} = u - \left(\frac{\partial u}{\partial x}\right)\frac{\mathrm{d}x}{2}.$$

Similar expressions for ρ and v for the front and back faces and ρ and w for the top and bottom faces of the infinitesimal control volume can be made. These can then be used to evaluate the surface integral in Equation (10) (i.e. we can find the net flux of mass out of the control volume). Equation (10) states

$$\frac{\partial}{\partial t}\int_{\mathrm{CV}} \rho \,\mathrm{d}V + \int_{\mathrm{CS}} \rho \mathbf{V}\cdot \mathrm{d}\mathbf{A} = 0.$$

If we assume that the velocity components u, v, and w are positive in the x, y, and z directions, respectively, that the area normal is positive out of the cube and that higher order terms can be neglected as we find the limit of $\mathrm{d}x$, $\mathrm{d}y$, and $\mathrm{d}z$ as they go to zero, then we can solve the surface integral to be:

$$\int_{\mathrm{CS}} \rho \mathbf{V}\cdot \mathrm{d}\mathbf{A} = \left(\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z}\right)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$$

To complete Equation (10) we need to evaluate the volume integral:

$$\frac{\partial}{\partial t}\int_{\mathrm{CV}} \rho \,\mathrm{d}V = \frac{\partial}{\partial t}\left(\rho \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\right) = \frac{\partial \rho}{\partial t}\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$$

Hence we obtain the differential form of the mass conservation law:

$$\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} + \frac{\partial \rho}{\partial t} = 0 \tag{14}$$

Equation (14) is often called the *continuity equation*.

Since ∇ is given by:

$$\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

then

$$\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} = \nabla \cdot \rho \mathbf{V}.$$

Therefore the conservation of mass from Equation (14) can also be written as:

$$\nabla \cdot \rho \mathbf{V} + \frac{\partial \rho}{\partial t} = 0 \tag{15}$$

When Equation (14) is written on vector form as in Equation (15) it may be applied in any coordinate system – even a cylindrical or polar one.

Two special cases of the continuity equation are worth a mention.

9.1.2 Incompressible Fluid

For an incompressible fluid ρ = constant and is therefore neither a function of space coordinate nor time. Therefore the continuity equation for an incompressible fluid simplifies to:

$$\nabla \cdot \mathbf{V} = 0.$$

Thus the velocity field, $\mathbf{V}(x, y, z, t)$ must satisfy $\nabla \cdot \mathbf{V} = 0$ for an incompressible fluid.

9.1.3 Steady Flow

For steady flow, all fluid properties, are, by definition, independent of time. This for steady flow the continuity equation simplifies to:

$$\nabla \cdot \rho \mathbf{V} = 0.$$

9.2 Stream Function for Two-Dimensional Incompressible Flow

Streamlines have previously been defined as being lines tangent to the velocity vectors in a flow at an instant:

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{\text{streamline}} = \frac{v}{u}.$$

We will now seek to develop a more formal definition of stream lines by introducing the *stream function* ψ. This function allows us to represent two entities – the velocity components $u(x, y, t)$ and $v(x, y, t)$ of a two-dimensional incompressible flow – with a single function $\psi(x, y, t)$.

We start with a two-dimensional version of the continuity equation for incompressible flow:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{16}$$

We now define the stream function by:

$$u \equiv \frac{\partial \psi}{\partial y} \qquad \text{and} \qquad v \equiv -\frac{\partial \psi}{\partial x}.$$

By doing this Equation (16) is automatically satisfied for any $\psi(x, y, t)$. Using the "naive" definition of streamlines introduced at the beginning of this section we can obtain an equation that is only valid along a streamline:

$$u\,\mathrm{d}y - v\,\mathrm{d}x = 0$$

or using the stream function

$$\frac{\partial \psi}{\partial x}\,\mathrm{d}x + \frac{\partial \psi}{\partial y}\,\mathrm{d}y = 0 \tag{17}$$

From a strictly mathematical point of view, at any instant in time t the variation in a function $\psi(x, y, t)$ in space (x, y) is given by;

$$\mathrm{d}\psi = \frac{\partial \psi}{\partial x}\,\mathrm{d}x + \frac{\partial \psi}{\partial y}\,\mathrm{d}y \tag{18}$$

Comparing Equation (17) and Equation (18) we see that along an instantaneous streamline $\mathrm{d}\psi = 0$, i.e. ψ is constant along a streamline. Therefore we can specify individual streamlines by their stream function values $\psi = 0, 1, 2, \ldots$. Furthermore, these ψ values can be used to obtain the volume flow rate between any two streamlines. Consider the streamlines shown on Figure 9.2. We can compute the volume flow rate between ψ_1 and ψ_2 by using line AB, BC, DE, or EF (remember there is no flow across a streamline).

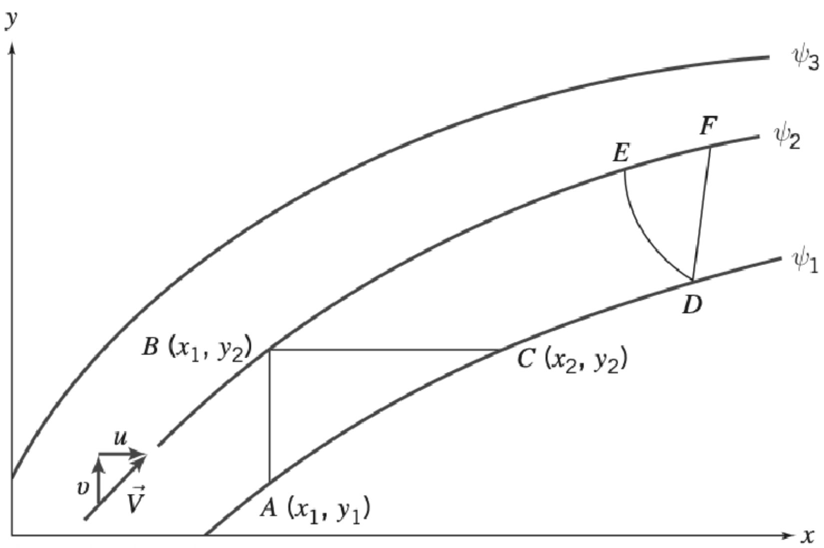


Figure 9.2: Instantaneous streamlines in a two-dimensional flow

Now let us compare the volume flow rate found using line AB and line BC – these should be the same.

For a unit depth, the flow rate across AB is

$$Q = \int_{y_1}^{y_2} u\,\mathrm{d}y = \int_{y_1}^{y_2} \frac{\partial \psi}{\partial y}\,\mathrm{d}y.$$

Along AB, $x = \text{constant}$, thus

$$\mathrm{d}\psi = \frac{\partial \psi}{\partial y}\,\mathrm{d}y$$

therefore:

$$Q = \int_{y_1}^{y_2} \frac{\partial \psi}{\partial y}\,\mathrm{d}y = \int_{\psi_1}^{\psi_2} \mathrm{d}\psi = \psi_2 - \psi_1.$$

The same procedure is followed for BC, however here y = constant, thus:

$$\begin{aligned} Q &= \int_{x_1}^{x_2} v \,\mathrm{d}x \\ &= -\int_{x_1}^{x_2} \frac{\partial \psi}{\partial x} \,\mathrm{d}x \\ &= -\int_{\psi_2}^{\psi_1} \mathrm{d}\psi = \psi_2 - \psi_1. \end{aligned}$$

Hence the volume flow rate per unit depth between two streamlines is given by the difference between the stream function values.

For two-dimensional steady compressible flow in the xy plane, the stream function ψ can be defined such that:

$$\rho u \equiv \frac{\partial \psi}{\partial y} \qquad \text{and} \qquad \rho v \equiv -\frac{\partial \psi}{\partial x}.$$

Then the difference between the constant values of ψ defining two streamlines is the mass flow rate per unit depth between the two streamlines.

If $\psi = 0$ is used to designate the streamline through the origin, the all other ψ values for all other streamlines represents the flow between the origin and that streamline.

Lecture 6: Differential Analysis II: Kinematics & Navier-Stokes Equations September 10, 2025

10.1 Motion of a Fluid Particle (Kinematics)

On Figure 10.1 a typical finite fluid element is shown. Within this an infinitesimal particle with mass $\mathrm{d}m$ and initial volume $\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$, at time t as well as how it may look after an infinitesimal time interval $\mathrm{d}t$. After this period the finite element has moved and changed its shape. We will examine the infinitesimal particle so that we eventually obtain results applicable to a point. We wan decompose this infinitesimal particle's motion into four components:

- translation, where the particle moves from one point to another
- rotation, which can occur about any or all of the x, y or z axes
- liner deformation, in which the particle's sides stretch or contract
- angular deformation, in which angles that were initially 90° between the sides of the particle changes

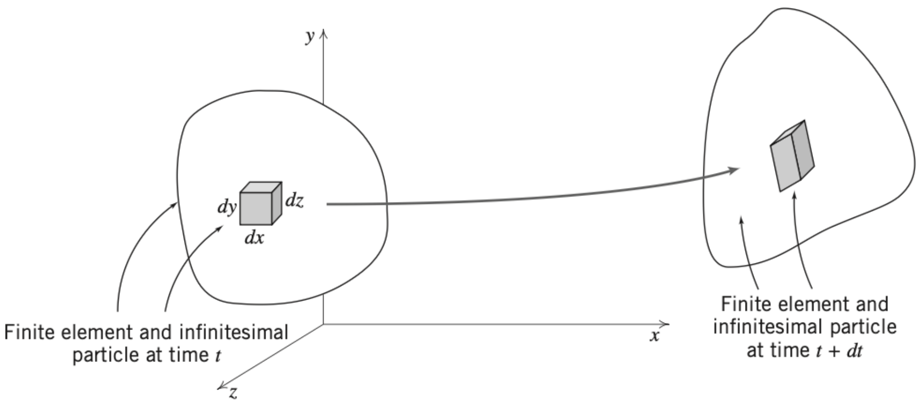


Figure 10.1: Finite fluid element and infinitesimal particle at times t and $t + \mathrm{d}t$

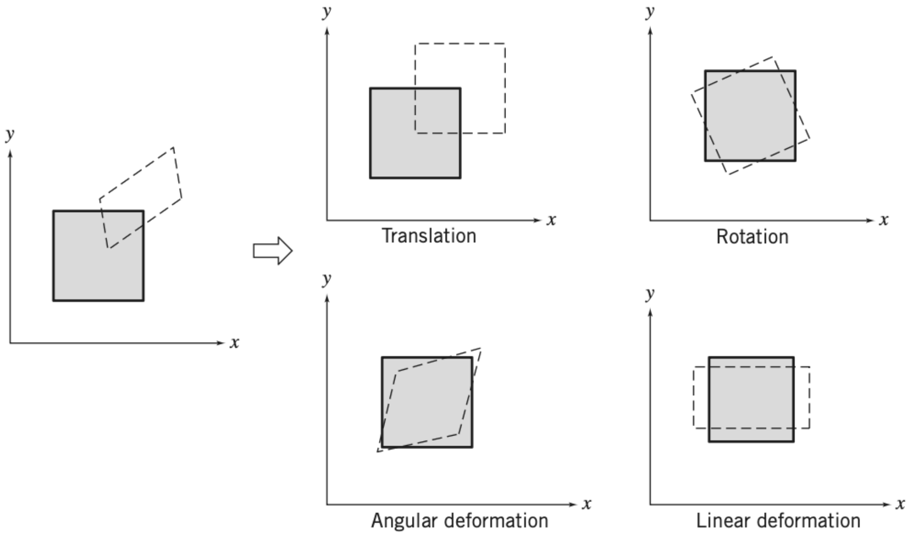


Figure 10.2: Pictorial representation of the components of fluid motion.

10.1.1 Fluid Translation: Acceleration of a Fluid Particle in a Velocity Field

We will need the acceleration of a fluid particle to be able to apply Newton's second law to it.

Consider a particle moving in a velocity field. At time, t, the particle is in position x, y, z and has a velocity vector corresponding to the velocity at that point in space at time t,

$$\mathbf{V}_P\Big)_t = \mathbf{V}(x, y, z, t).$$

At time $t + \mathrm{d}t$ the particle has moved to a new position with coordinates $x + \mathrm{d}x, y + \mathrm{d}y, z + \mathrm{d}z$ and its velocity will when be given by

$$\mathbf{V}_P\Big)_{t+\mathrm{d}t} = \mathbf{V}\left(x + \mathrm{d}x, y + \mathrm{d}y, z + \mathrm{d}z, t + \mathrm{d}t\right).$$

If the particle velocity at time t (position $\mathbf{r}$) is given by $\mathbf{V}_P = \mathbf{V}(x, y, z, t)$, then $\mathrm{d}\mathbf{V}_p$ is given by the chain rule as:

$$\mathrm{d}\mathbf{V}_p = \frac{\partial \mathbf{V}}{\partial x}\mathrm{d}x_p + \frac{\partial \mathbf{V}}{\partial y}\mathrm{d}y_p + \frac{\partial \mathbf{V}}{\partial z}\mathrm{d}z_p + \frac{\partial \mathbf{V}}{\partial t}\mathrm{d}t.$$

The total acceleration of the particle is given by:

$$\mathbf{a}_p = \frac{\mathrm{d}\mathbf{V}_p}{\mathrm{d}t} = \frac{\partial \mathbf{V}}{\partial x}\frac{\mathrm{d}x_p}{\mathrm{d}t} + \frac{\partial \mathbf{V}}{\partial y}\frac{\mathrm{d}y_p}{\mathrm{d}t} + \frac{\partial \mathbf{V}}{\partial z}\frac{\mathrm{d}z_p}{\mathrm{d}t} + \frac{\partial \mathbf{V}}{\partial t}.$$

And since

$$\frac{\mathrm{d}x_p}{\mathrm{d}t} = u, \quad \frac{\mathrm{d}y_p}{\mathrm{d}t} = v, \quad \frac{\mathrm{d}z_p}{\mathrm{d}t} = w$$

we have

$$\mathbf{a}_p = \frac{\mathrm{d}\mathbf{V}_p}{\mathrm{d}t} = u\frac{\partial \mathbf{V}}{\partial x} + v\frac{\partial \mathbf{V}}{\partial y} + w\frac{\partial \mathbf{V}}{\partial z} + \frac{\partial \mathbf{V}}{\partial t}.$$

A special symbol $\frac{\mathrm{D}\mathbf{V}}{\mathrm{D}t}$ is given to the derivative to remind us that it is not simply a normal derivative, thus:

$$\frac{\mathrm{D}\mathbf{V}}{\mathrm{D}t} \equiv \mathbf{a}_p = u\frac{\partial \mathbf{V}}{\partial x} + v\frac{\partial \mathbf{V}}{\partial y} + w\frac{\partial \mathbf{V}}{\partial z} + \frac{\partial \mathbf{V}}{\partial t}.$$

This derivative is sometimes called the *substantial derivative*, the *material derivative* or the *particle derivative*.

From this we see that a fluid particle moving in a flow field may accelerate for either of two reasons. Either through *convective acceleration*, where fluid particles are *convected* between regions with different velocities. If the flow field is unsteady an additional *local acceleration* is applied, because the velocity field is then a function of time.

The convective acceleration can be written using the gradient operator as:

$$u\frac{\partial \mathbf{V}}{\partial x}+v\frac{\partial \mathbf{V}}{\partial y}+w\frac{\partial \mathbf{V}}{\partial z}=(\mathbf{V}\cdot\nabla)\,\mathbf{V}.$$

Thus the acceleration may be written as:

$$\frac{\mathrm{D}\mathbf{V}}{\mathrm{D}t}\equiv\mathbf{a}_p=(\mathbf{V}\cdot\nabla)\,\mathbf{V}+\frac{\partial \mathbf{V}}{\partial t}.$$

For a two dimensional flow, this becomes:

$$\frac{\mathrm{D}\mathbf{V}}{\mathrm{D}t}=u\frac{\partial \mathbf{V}}{\partial x}+v\frac{\partial \mathbf{V}}{\partial y}+\frac{\partial \mathbf{V}}{\partial t}.$$

For a one dimensional flow it becomes:

$$\frac{\mathrm{D}\mathbf{V}}{\mathrm{D}t}=u\frac{\partial \mathbf{V}}{\partial x}+\frac{\partial \mathbf{V}}{\partial t}.$$

And finally for a steady flow in three dimensions it becomes:

$$\frac{\mathrm{D}\mathbf{V}}{\mathrm{D}t}=u\frac{\partial \mathbf{V}}{\partial x}+v\frac{\partial \mathbf{V}}{\partial y}+w\frac{\partial \mathbf{V}}{\partial z}.$$

This means that a fluid particle may undergo convective acceleration even in a steady velocity field simply due to its motion. As scalar component equation we have

$$\begin{aligned}
a_{x_p}&=\frac{\mathrm{D}u}{\mathrm{D}t}=u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}+\frac{\partial u}{\partial t}\\
a_{y_p}&=\frac{\mathrm{D}v}{\mathrm{D}t}=u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}+\frac{\partial v}{\partial t}\\
a_{z_p}&=\frac{\mathrm{D}w}{\mathrm{D}t}=u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}+\frac{\partial w}{\partial t}.
\end{aligned}$$

10.1.2 Fluid Rotation

The fluid particle may rotate about all coordinate axes. In general,

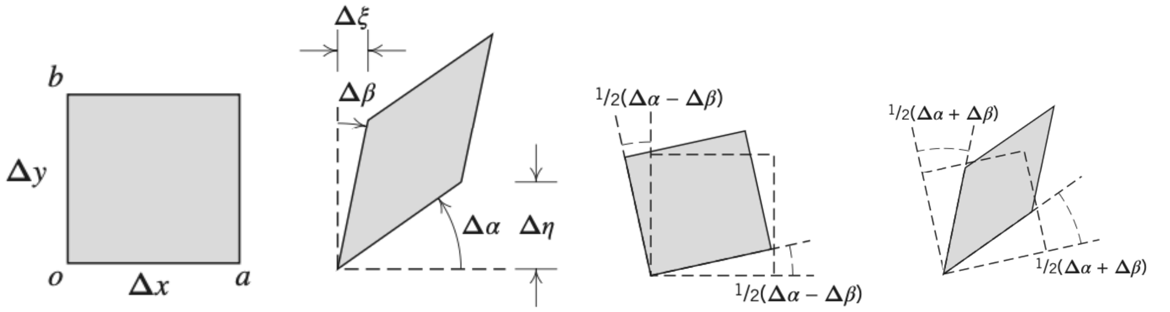
$$\boldsymbol{\omega}=\hat{i}\omega_x+\hat{j}\omega_y+\hat{k}\omega_z.$$

We now consider the xy plane view of the particle at time t on Figure 10.3. The left and lower sides of the particle are given by the two perpendicular line segments oa and ob of length Δx and Δy, respectively. After a time interval Δt the particle will have translated to some new position and also have rotated and deformed.

These two sides may now have deformed by the same angle $\Delta\alpha$ and $\Delta\beta$. To extract a rotation from this one takes the average of the rotations of $\Delta\alpha$ and $\Delta\beta$ such that the counterclockwise rotation is $\frac{1}{2}\left(\Delta\alpha-\Delta\beta\right)$.

From $\Delta\alpha$ and $\Delta\beta$ we can also determine a measure of the particle's angular deformation. To obtain the deformation of side oa we subtract the particle rotation $\frac{1}{2}\left(\Delta\alpha-\Delta\beta\right)$ from the actual rotation of oa, $\Delta\alpha$, what remains is the deformation of side oa. The total deformation of the particle is the sum of the deformations of the sides or $\Delta\alpha+\Delta\beta$.

Now we need to convert these angular measures to quantities obtainable from the flow field. To do this, we



(a) Original particle (b) Particle after time Δt (c) Rotational component (d) Angular deformation component

Figure 10.3: Rotation and angular deformation of perpendicular line segments in a two-dimensional flow.

recognize that for small angles $\Delta\alpha = \frac{\Delta\eta}{\Delta x}$, and $\Delta\beta = \frac{\Delta\xi}{\Delta y}$. $\Delta\xi$ arises because, if in an interval Δt point o moves horizontally distance $u\Delta t$ then point b will have moved a distance $\left(u + \left(\frac{\partial u}{\partial y}\right)\Delta y\right)\Delta t$. Likewise for $\Delta\eta$. Hence,

$$\Delta\xi = \left(u + \frac{\partial u}{\partial y}\Delta y\right)\Delta t - u\Delta t = \frac{\partial u}{\partial y}\Delta y \Delta t$$
$$\Delta\eta = \left(v + \frac{\partial v}{\partial x}\Delta x\right)\Delta t - v\Delta t = \frac{\partial v}{\partial x}\Delta x \Delta t.$$

We can now compute the angular velocity of the particle about the z axis ω_z by combining all these results:

$$\begin{aligned}
\omega_z &= \lim_{\Delta t \to 0} \frac{\frac{1}{2}\left(\Delta\alpha - \Delta\beta\right)}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{\frac{1}{2}\left(\frac{\Delta\eta}{\Delta x} - \frac{\Delta\xi}{\Delta y}\right)}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{\frac{1}{2}\left(\frac{\partial v}{\partial x}\frac{\Delta x}{\Delta x}\Delta t - \frac{\partial u}{\partial y}\frac{\Delta y}{\Delta y}\Delta t\right)}{\Delta t} \\
&= \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right).
\end{aligned}$$

Similarly one can show that:

$$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)$$
$$\omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right).$$

Then $\boldsymbol{\omega}$ becomes:

$$\boldsymbol{\omega} = \frac{1}{2}\left(\hat{i}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) + \hat{j}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + \hat{k}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\right).$$

The term in the outermost parenthesis can be written

$$\text{curl}\mathbf{V} = \nabla \times \mathbf{V}.$$

And therefore we get:

$$\boldsymbol{\omega} = \frac{1}{2}\nabla \times \mathbf{V}.$$

Fluid rotation can arise in a number of different ways. One way, if the particles are not rotating initially, for rotation to be induced is due to torque caused by surface shear stresses – the particle body force and pressure forces cannot generate torque. Hence rotation of fluid particles will always occur when we have shear stresses.

Flows without particle rotation are called *irrotational flows*. Although no real flow is truly irrotational as all fluids are viscous, many flows can be studied assuming they are inviscid and irrotational.

The factor of $\frac{1}{2}$ can be eliminated in the expression for $\boldsymbol{\omega}$ by defining the vorticity $\boldsymbol{\xi}$ to be twice the rotation,

$$\boldsymbol{\xi} \equiv 2\boldsymbol{\omega} = \nabla \times \mathbf{V}.$$

The *circulation*, Γ, is defined as the line integral of the tangential velocity component about any closed curve fixed in the flow,

$$\Gamma = \oint_c \mathbf{V} \cdot \mathrm{d}\mathbf{s}$$

where $\mathrm{d}\mathbf{s}$ is an elemental vector tangent to the curve with length $\mathrm{d}s$ of the element of arc. We can develop a relationship between circulation and vorticity by considering the rectangular circuit shown in Figure 10.4, where the velocity components at o are set to (u, v) and the velocities along segments bc and ac can be derived using Taylor series approximations.

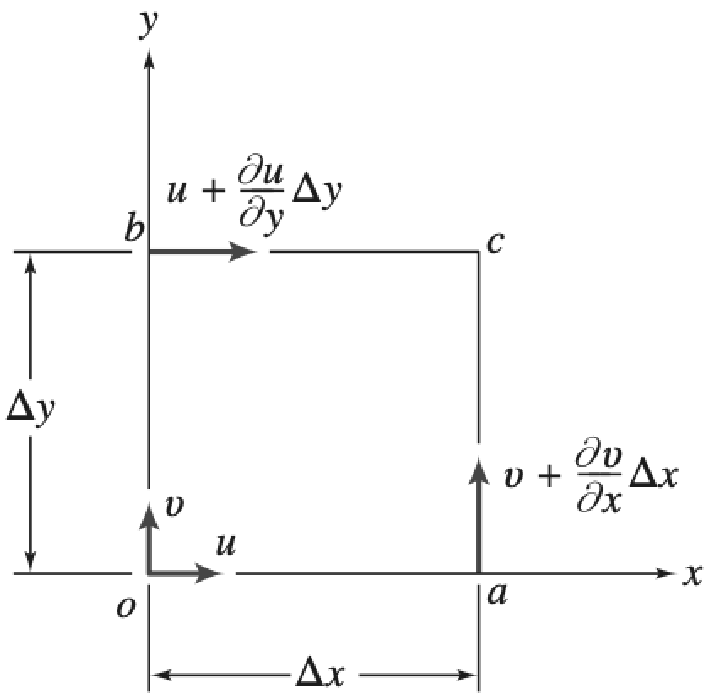


Figure 10.4: Velocity components on the boundaries of a fluid element.

For the closed curve $oacb$ we get

$$\begin{aligned}
\Delta\Gamma &= u\Delta x + \left(v + \frac{\partial v}{\partial x}\Delta x\right)\Delta y - \left(u + \frac{\partial u}{\partial y}\Delta y\right)\Delta x - v\Delta y \\
&= \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\Delta x \Delta y \\
&= 2\omega_z \Delta x \Delta y.
\end{aligned}$$

Then

$$\Gamma = \oint_c \mathbf{V} \cdot \mathrm{d}\mathbf{s} = \int_A 2\omega_z \,\mathrm{d}A = \int_A (\nabla \times \mathbf{V})_z \,\mathrm{d}A \tag{19}$$

Equation (19) is a statement of the Stokes theorem in two dimensions; the circulation around a closed contour is equal to the total vorticity enclosed within it.

10.1.3 Fluid Deformation

Angular Deformation The total angular deformation of a particle is given by the sum of the two angular deformations, i.e. $\Delta\alpha + \Delta\beta$. We have also found $\Delta\alpha = \frac{\Delta\eta}{\Delta x}$, $\Delta\beta = \frac{\Delta\xi}{\Delta y}$ where $\Delta\xi$ and $\Delta\eta$ are given by

$$\Delta\xi = \left(u + \frac{\partial u}{\partial y}\Delta y\right)\Delta t - u\Delta t = \frac{\partial u}{\partial y}\Delta y\Delta t$$
$$\Delta\eta = \left(v + \frac{\partial v}{\partial x}\Delta x\right)\Delta t - v\Delta t = \frac{\partial v}{\partial x}\Delta x\Delta t.$$

By combining these results we can compute the rate of angular deformation of the particle in the xy as

$$\begin{aligned}\text{Rate of angular deformation}_{xy} &= \lim_{\Delta t\to 0}\frac{(\Delta\alpha + \Delta\beta)}{\Delta t}\\ &= \lim_{\Delta t\to 0}\frac{\frac{\Delta\eta}{\Delta x} + \frac{\Delta\xi}{\Delta y}}{\Delta t}\\ &= \lim_{\Delta t\to 0}\frac{\frac{\partial v}{\partial x}\frac{\Delta x}{\Delta x}\Delta t + \frac{\partial u}{\partial y}\frac{\Delta y}{\Delta y}\Delta t}{\Delta t}\\ &= \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}.\end{aligned}$$

Similar expressions can be found for the angular deformation in the yz and zx planes:

$$\text{Rate of angular deformation}_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}$$
$$\text{Rate of angular deformation}_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}.$$

We have previously shown that for one-dimensional laminar Newtonian flow the shear stress is given by the rate of deformation of the particle:

$$\tau_{yx} = \mu\frac{\mathrm{d}u}{\mathrm{d}y}.$$

This is a special case of the expression derived above.

Linear Deformation During linear deformation, the shape of the fluid element, described by the angles at its vertices, is unchanged. The element will change length in the x direction only of $\frac{\partial u}{\partial x} \neq 0$, similarly a change in the y and z directions only happens for $\frac{\partial v}{\partial y} \neq 0$ and $\frac{\partial w}{\partial z} \neq 0$ respectively.

The rate of local instantaneous *volume dilation* is

$$\text{Volume dilation rate} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \nabla\cdot\mathbf{V}.$$

For incompressible flow, the rate of volume dilation is zero.

10.2 Momentum Equation

A dynamic equation describing the motion of a fluid may be obtained by applying Newton's second law to a particle. To derive the differential form of the momentum equation, we shall apply Newton's second law to an infinitesimal fluid particle of mass $\mathrm{d}m$.

Newton's second law for a finite system is

$$\mathbf{F} = \left.\frac{\mathrm{d}\mathbf{P}}{\mathrm{d}t}\right)_{\text{system}}$$

where the linear momentum $\mathbf{P}$ of the system is given by

$$\mathbf{P}_{\text{system}} = \int_{\text{mass(system)}} \mathbf{V}\,\mathrm{d}m.$$

For an infinitesimal system of mass $\mathrm{d}m$ this law can be written as:

$$\mathrm{d}\mathbf{F} = \mathrm{d}m \left.\frac{\mathrm{d}\mathbf{V}}{\mathrm{d}t}\right)_{\text{system}}.$$

As we have an expression for the acceleration of a fluid element of mass $\mathrm{d}m$ moving in a velocity field we can write Newton's second law as the vector equation:

$$\mathrm{d}\mathbf{F} = \mathrm{d}m \frac{\mathrm{D}\mathbf{V}}{\mathrm{D}t} = \mathrm{d}m \left(u\frac{\partial \mathbf{V}}{\partial x} + v\frac{\partial \mathbf{V}}{\partial y} + w\frac{\partial \mathbf{V}}{\partial z} + \frac{\partial \mathbf{V}}{\partial t} \right).$$

Now we need a formulation for the force $\mathrm{d}\mathbf{F}$ acting on the element.

10.2.1 Forces Acting on a Fluid Particle

We shall consider the x component of the force acting on a differential element of mass $\mathrm{d}m$ and volume $\mathrm{d}V = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$. Only stresses acting in the x direction will give rise to surface forces in the x direction. If the stresses at the center of the differential element are taken to be σ_{xx}, τ_{yx}, and τ_{zx}, then the stresses acting on the x direction can be obtained using a Taylor series expansion about the center of the element. The result of this is shown on Figure 10.5

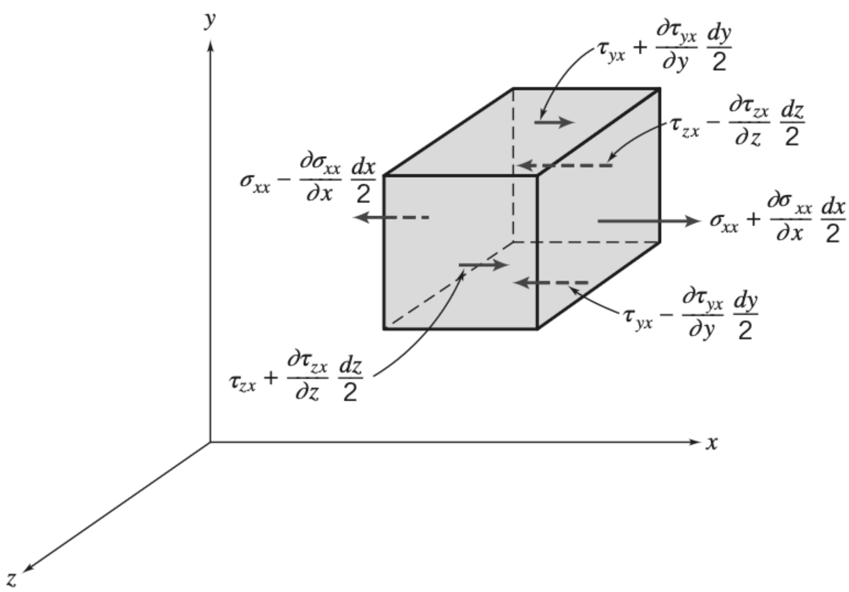


Figure 10.5: Stresses in the x direction on a fluid element.

To obtain the net surface force in the x direction $\mathrm{d}F_{S_x}$ we must sum the forces in the x-direction. On simplifying this we obtain:

$$\mathrm{d}F_{S_x} = \left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \right) \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$$

Assuming the gravitational force is the only body force acting on the system, then the net force in the x direction $\mathrm{d}F_x$ is given by:

$$\mathrm{d}F_x = \mathrm{d}F_{B_x} + \mathrm{d}F_{S_x} = \left(\rho g_x + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \right) \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$$

Similarly for the force in the y and z directions:

$$\mathrm{d}F_y = \mathrm{d}F_{B_y} + \mathrm{d}F_{S_y} = \left(\rho g_y + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}\right) \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$
$$\mathrm{d}F_z = \mathrm{d}F_{B_z} + \mathrm{d}F_{S_z} = \left(\rho g_z + \frac{\partial \tau xz}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z}\right) \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$$

10.2.2 Differential Momentum Equation

Now we have defined the force components acting on a fluid element of mass $\mathrm{d}m$. If we substitute these expressions into Newton's second law from before we obtain the differential equations of motion,

$$\rho g_x + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right)$$
$$\rho g_y + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} = \rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right)$$
$$\rho g_z + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} = \rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right).$$

These are the differential equations of motion for any fluid satisfying the continuum assumption.

10.2.3 Newtonian Fluid: Navier-Stokes Equations

For a Newtonian fluid the viscous stress is directly proportional to the rate of shearing strain (the angular deformation rate). The stresses may be expressed in terms of velocity gradients and fluid properties in rectangular coordinates as:

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$$
$$\tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)$$
$$\tau_{zx} = \tau_{xz} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$
$$\sigma_{xx} = -p - \frac{2}{3}\mu\nabla\mathbf{V} + 2\mu\frac{\partial u}{\partial x}$$
$$\sigma_{yy} = -p - \frac{2}{3}\mu\nabla\mathbf{V} + 2\mu\frac{\partial v}{\partial y}$$
$$\sigma_{zz} = -p - \frac{2}{3}\mu\nabla\cdot\mathbf{V} + 2\mu\frac{\partial w}{\partial z}.$$

There p is the local pressure. If these expressions for the stresses are introduced into the differential equations of motion we obtain:

$$\rho\frac{\mathrm{D}u}{\mathrm{D}t} = \rho g_x - \frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left(\mu\left(2\frac{\partial u}{\partial x} - \frac{2}{3}\nabla\cdot\mathbf{V}\right)\right) + \frac{\partial}{\partial y}\left(\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right) + \frac{\partial}{\partial z}\left(\mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\right)$$
$$\rho\frac{\mathrm{D}v}{\mathrm{D}t} = \rho g_y - \frac{\partial p}{\partial y} + \frac{\partial}{\partial x}\left(\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right) + \frac{\partial}{\partial y}\left(\mu\left(2\frac{\partial v}{\partial y} - \frac{2}{3}\nabla\cdot\mathbf{V}\right)\right) + \frac{\partial}{\partial z}\left(\mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right)$$
$$\rho\frac{\mathrm{D}w}{\mathrm{D}t} = \rho g_z - \frac{\partial p}{\partial z} + \frac{\partial}{\partial x}\left(\mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\right) + \frac{\partial}{\partial y}\left(\mu\left(\frac{\partial w}{\partial z} + \frac{\partial w}{\partial y}\right)\right) + \frac{\partial}{\partial z}\left(\mu\left(2\frac{\partial w}{\partial z} - \frac{2}{3}\nabla\cdot\mathbf{V}\right)\right).$$

These equations of motion are called the *Navier-Stokes equations*. For an incompressible flow these reduce to:

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=\rho g_x-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)$$
$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)=\rho g_y-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)$$
$$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=\rho g_z-\frac{\partial p}{\partial z}+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right).$$

These describe many common flows so long as the fluids are Newtonian and incompressible.

Lecture 7: Euler and Bernoulli Equations September 15, 2025

12 Incompressible Inviscid Flow

The Navier-Stokes equations introduced in Section 10.2.3 describe fluid behaviour in a wide range of problems. However these are not analytically solvable except in simple cases. Instead of the Navier-Stokes equations we can instead use Euler's equation, which applies to an inviscid ($\mu = 0$) fluid (which itself is an idealized notion as no flow is truly inviscid).

12.1 Momentum Equation for Frictionless Flow: Euler's Equation

Euler's equation can be obtained from the Navier-Stokes equations simply by neglecting the viscous term. I.e.

$$\rho\frac{\mathrm{D}\mathbf{V}}{\mathrm{D}t}=\rho\mathbf{g}-\nabla p \tag{20}$$

This states that for an inviscid flow, the change in momentum of a fluid particle is caused by the body force and the net pressure force. In rectangular coordinates this is

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=\rho g_x-\frac{\partial p}{\partial x}$$
$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)=\rho g_y-\frac{\partial p}{\partial y}$$
$$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=\rho g_z-\frac{\partial p}{\partial w}.$$

If the z-axis is vertical, then $\mathbf{g}=-g\hat{k} \implies g_x=g_y=0, g_z=-g$.

In general, Equation (20) is applicable for any problem in which there is no viscous stresses. This also includes rigid-body motion problems.

We consider the flow in the yz-plane shown on Figure 12.1. Here our goal is to write the equations of motion in terms of the distance along the streamline s and the distance normal to the streamline n. The pressure at the center of the fluid element is assumed to be p. If we apply Newton's second law in the direction s of the streamline, to the fluid element on Figure 12.1 with volume $\mathrm{d}s\,\mathrm{d}n\,\mathrm{d}x$, neglecting viscous forces we obtain

$$\left(p-\frac{\partial p}{\partial s}\frac{\mathrm{d}s}{2}\right)\mathrm{d}n\,\mathrm{d}x-\left(p+\frac{\partial p}{\partial s}\frac{\mathrm{d}s}{2}\right)\mathrm{d}n\,\mathrm{d}x-\rho g\sin\beta\,\mathrm{d}s\,\mathrm{d}n\,\mathrm{d}x=\rho a_s\,\mathrm{d}s\,\mathrm{d}n\,\mathrm{d}x$$

where β is the angle between the tangent to the streamline and the horizontal, and a_s is the acceleration of the fluid particle along the streamline. By simplification, we obtain

$$-\frac{\partial p}{\partial s}-\rho g\sin\beta=\rho a_s.$$

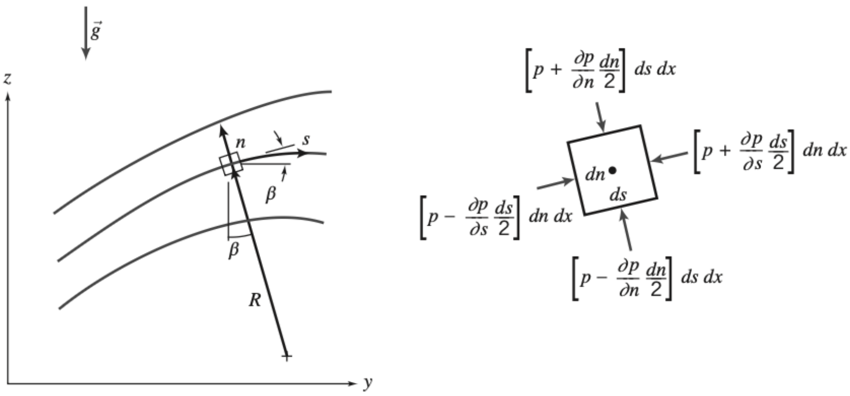


Figure 12.1: Fluid particle moving along a streamline.

Since $\sin\beta = \frac{\partial z}{\partial s}$, this can be written as

$$-\frac{1}{\rho}\frac{\partial p}{\partial s} - g\frac{\partial z}{\partial s} = a_s$$

which holds along any streamline $V = V(s,t)$. We remember that the total acceleration of a fluid particle in the streamwise direction is

$$a_s = \frac{\mathrm{D}V}{\mathrm{D}t} = \frac{\partial V}{\partial t} + V\frac{\partial V}{\partial s}.$$

Hence Euler's equation in the streamwise direction with the z-axis directed vertically upward is

$$-\frac{1}{\rho}\frac{\partial p}{\partial s} - g\frac{\partial z}{\partial s} = \frac{\partial V}{\partial t} + V\frac{\partial V}{\partial s}.$$

For steady flow this reduces to

$$\frac{1}{\rho}\frac{\partial p}{\partial s} = -g\frac{\partial z}{\partial s} - V\frac{\partial V}{\partial s} \tag{21}$$

This indicates that the pressure along a streamline is influenced by the gravitational field and the velocity. The gravitational effect has already been described in Section 4 – When $V = 0$, the pressure increases directly proportionally with the change in elevation. In the xy-plane, in which there is no influence of gravity Equation (21) reduces to

$$\frac{1}{p}\frac{\partial p}{\partial s} = -V\frac{\partial V}{\partial s}$$

which states that for an incompressible, inviscid flow a decrease in pressure will cause an increased velocity and v.v.

To obtain Euler's equation in a direction normal to the streamlines, we apply Newton's second law in the n direction to the fluid element. Again, neglecting viscous forces, we get

$$\left(p - \frac{\partial p}{\partial n}\frac{\mathrm{d}n}{2}\right)\mathrm{d}s\,\mathrm{d}x - \left(p + \frac{\partial p}{\partial n}\frac{\mathrm{d}n}{2}\right)\mathrm{d}s\,\mathrm{d}x - \rho g\cos\beta\,\mathrm{d}n\,\mathrm{d}x\,\mathrm{d}s = \rho a_n\,\mathrm{d}n\,\mathrm{d}x\,\mathrm{d}s$$

where β is the angle between the n-direction and the vertical and a_n is the acceleration of the fluid particle in the n-direction On simplification this becomes

$$-\frac{\partial p}{\partial n} - \rho g\cos\beta = \rho a_n.$$

And since $\cos\beta = \frac{\partial z}{\partial n}$ this can be written as

$$-\frac{1}{\rho}\frac{\partial p}{\partial n} - g\frac{\partial z}{\partial n} = a_n.$$

The normal acceleration of the fluid element is toward the center of curvature of the streamline – in the $-n$-direction. Hence

$$a_n = -\frac{V^2}{R}.$$

Therefore, for steady flow, where R is the radius of curvature of the streamline at the point chosen, Euler's equation normal to the streamline is

$$\frac{1}{\rho}\frac{\partial p}{\partial n} + g\frac{\partial z}{\partial n} = \frac{V^2}{R}.$$

In a horizontal plane this becomes

$$\frac{1}{\rho}\frac{\partial p}{\partial n} = \frac{V^2}{R}.$$

12.2 Bernoulli Equation: Integration of Euler's Equation Along a Streamline for Steady Flow

Compared to the Navier-Stokes equations the Euler's equation for incompressible inviscid flow is mathematically simpler – however solving it still presents difficulty in complex cases. One convenient approach to circumvent this for a steady flow is to integrate Euler's equation along a streamline.

12.2.1 Derivation of the Bernoulli Equation Using Streamline Coordinates

Euler's equation for steady flow along a streamline is given in Equation (21) as

$$-\frac{1}{\rho}\frac{\partial p}{\partial s} - g\frac{\partial z}{\partial s} = V\frac{\partial V}{\partial s}.$$

If a fluid particle moves an infinitesimal distance ds along a streamline, then

$$\begin{aligned}
\frac{\partial p}{\partial s}\,\mathrm{d}s &= \mathrm{d}p && \text{(The change in pressure along } s\text{)}\\
\frac{\partial z}{\partial s}\,\mathrm{d}s &= \mathrm{d}z && \text{(The change in elevation along } s\text{)}\\
\frac{\partial V}{\partial s}\,\mathrm{d}V &= \mathrm{d}p && \text{(The change in speed along } s\text{).}
\end{aligned}$$

Thus, by multiplying Equation (21) by ds we get

$$-\frac{\mathrm{d}p}{\rho} - g\,\mathrm{d}z = V\,\mathrm{d}V \implies \frac{\mathrm{d}p}{\rho} + V\,\mathrm{d}V + g\,\mathrm{d}z = 0 \qquad \text{(along } s\text{).}$$

Integration of this equation gives

$$\int\frac{\mathrm{d}p}{\rho} + \frac{V^2}{2} + gz = \text{constant} \qquad \text{(along } s\text{).}$$

To use this one must first find the inverse change in density with pressure $\frac{1}{\rho}\,\mathrm{d}p$. For incompressible flow $\rho = \text{constant}$ so the above equation becomes the Bernoulli equation,

$$\frac{p}{\rho} + \frac{V^2}{2} + gz = \text{constant} \tag{22}$$

For Equation (22) to be applicable the flow must be steady, incompressible, frictionless and happen along a streamline.

12.2.2 Derivation of the Bernoulli Equation Using Rectangular Coordinates

Instead of using Euler's equation on streamline-coordinate form as in Section 12.2.1 one could also integrate the vector form of Euler's equation from Equation (20) along a streamline. As we are simply interested in the Bernoulli equation we will only consider the case of steady flow.

For steady flow, Euler's equation in rectangular coordinates is

$$\frac{\mathrm{D}\mathbf{V}}{\mathrm{D}t} = u\frac{\partial \mathbf{V}}{\partial x} + v\frac{\partial \mathbf{V}}{\partial y} + w\frac{\partial \mathbf{V}}{\partial z} = (\mathbf{V}\cdot\nabla)\,\mathbf{V} = -\frac{1}{\rho}\nabla p - g\hat{k} \tag{23}$$

For steady flow the velocity field is $\mathbf{V} = \mathbf{V}(x, y, z)$. The motion of a particle along a streamline is governed by Equation (23). During the time interval $\mathrm{d}t$ the particle has a displacement $\mathrm{d}\mathbf{s}$ along the streamline.

If we take the dot product of Equation (23) and the displacement $\mathrm{d}\mathbf{s}$ along the streamline, a scalar equation relating pressure, speed, and elevation is obtained along the streamline. I.e.

$$(\mathbf{V}\cdot\nabla)\,\mathbf{V}\cdot\mathrm{d}\mathbf{s} = -\frac{1}{\rho}\nabla p\cdot\mathrm{d}\mathbf{s} - g\hat{k}\cdot\mathrm{d}\mathbf{s}$$

where

$$\mathrm{d}\mathbf{s} = \mathrm{d}x\,\hat{\imath} + \mathrm{d}y\,\hat{\jmath} + \mathrm{d}z\,\hat{k} \qquad (\text{along } s).$$

Now we can evaluate each of the three terms one-by-one, starting with $-\frac{1}{\rho}\nabla p\cdot\mathrm{d}\mathbf{s}$,

$$\begin{aligned}
-\frac{1}{\rho}\nabla p\cdot\mathrm{d}\mathbf{s} &= -\frac{1}{\rho}\left(\hat{\imath}\frac{\partial p}{\partial x} + \hat{\jmath}\frac{\partial p}{\partial y} + \hat{k}\frac{\partial p}{\partial z}\right)\cdot\left(\mathrm{d}x\,\hat{\imath} + \mathrm{d}y\,\hat{\jmath} + \mathrm{d}z\,\hat{k}\right) && (\text{along } s)\\
&= -\frac{1}{\rho}\left(\frac{\partial p}{\partial x}\mathrm{d}x + \frac{\partial p}{\partial y}\mathrm{d}y + \frac{\partial p}{\partial z}\mathrm{d}z\right) && (\text{along } s)\\
&= -\frac{1}{\rho}\mathrm{d}p && (\text{along } s).
\end{aligned}$$

Moving on to $-g\hat{k}\cdot\mathrm{d}\mathbf{s}$, we get

$$\begin{aligned}
-g\hat{k}\cdot\mathrm{d}\mathbf{s} &= -g\hat{k}\cdot\left(\mathrm{d}x\,\hat{\imath} + \mathrm{d}y\,\hat{\jmath} + \mathrm{d}z\,\hat{k}\right) && (\text{along } s)\\
&= -g\,\mathrm{d}z && (\text{along } s).
\end{aligned}$$

Now we just need to evaluate $(\mathbf{V}\cdot\nabla)\,\mathbf{V}\cdot\mathrm{d}\mathbf{s}$. A common vector identity allows us to rewrite this as

$$\begin{aligned}
(\mathbf{V}\cdot\nabla)\,\mathbf{V}\cdot\mathrm{d}\mathbf{s} &= \left(\frac{1}{2}\nabla(\mathbf{V}\cdot\mathbf{V}) - \mathbf{V}\times(\nabla\times\mathbf{V})\right)\cdot\mathrm{d}\mathbf{s}\\
&= \left(\frac{1}{2}\nabla(\mathbf{V}\cdot\mathbf{V})\right)\cdot\mathrm{d}\mathbf{s} - (\mathbf{V}\times(\nabla\times\mathbf{V}))\cdot\mathrm{d}\mathbf{s}.
\end{aligned}$$

The last term in this is zero, since $\mathbf{V}$ is parallel to $\mathrm{d}\mathbf{s}$. Hence,

$$\begin{aligned}
(\mathbf{V}\cdot\nabla)\,\mathbf{V}\cdot\mathrm{d}\mathbf{s} &= \frac{1}{2}\nabla(\mathbf{V}\cdot\mathbf{V})\cdot\mathrm{d}\mathbf{s} && (\text{along } s)\\
&= \frac{1}{2}\nabla\left(V^2\right)\cdot\mathrm{d}\mathbf{s} && (\text{along } s)\\
&= \frac{1}{2}\left(\hat{\imath}\frac{\partial V^2}{\partial x} + \hat{\jmath}\frac{\partial V^2}{\partial y} + \hat{k}\frac{\partial V^2}{\partial z}\right)\cdot\left(\mathrm{d}x\,\hat{\imath} + \mathrm{d}y\,\hat{\jmath} + \mathrm{d}z\,\hat{k}\right) && (\text{along } s)\\
&= \frac{1}{2}\left(\frac{\partial V^2}{\partial x}\mathrm{d}x + \frac{\partial V^2}{\partial y}\mathrm{d}y + \frac{\partial V^2}{\partial z}\mathrm{d}z\right) && (\text{along } s)\\
&= \frac{1}{2}d\left(V^2\right) && (\text{along } s).
\end{aligned}$$

Substituting these three evaluated terms into the expression yields

$$\frac{\mathrm{d}p}{\rho} + \frac{1}{2}d\left(V^2\right) + g\mathrm{d}z = 0 \qquad \text{(along } s\text{)}.$$

Integrating this we obtain

$$\int \frac{\mathrm{d}p}{\rho} + \frac{V^2}{2} + gz = \text{constant}.$$

And for constant density this becomes the Bernoulli equation, Equation (22),

$$\frac{p}{\rho} + \frac{V^2}{2} + gz = \text{constant}.$$

Once again, for Equation (22) to be applicable the flow must be steady, incompressible, frictionless and happen along a streamline.

12.2.3 Static, Stagnation, and Dynamic Pressures

The pressure p which is used in the Bernoulli equation, Equation (22) is the thermodynamic pressure, also called the *static pressure*. This is the pressure measured by an observer riding along with the fluid.

The *stagnation pressure* is obtained when a flowing fluid is decelerated to zero speed by a frictionless process. For incompressible flow, the Bernoulli equation can be used to relate changes in speed and pressure along a streamline for such a process. Neglecting elevation differences Equation (22) becomes

$$\frac{p}{\rho} + \frac{V^2}{2} = \text{constant}.$$

The stagnation pressure p_0 can then be computed by setting the stagnation speed $V_0 = 0$, as

$$\frac{p_0}{\rho} + 0 = \frac{p}{\rho} + \frac{V^2}{2} \implies p_0 = p + \frac{1}{2}\rho V^2 \tag{24}$$

This is the mathematical definition of stagnation pressure for incompressible flow. The term $\frac{1}{2}\rho V^2$ is called the *dynamic pressure*. Thus, the stagnation (or *total*) pressure is the sum of the static pressure and the dynamic pressure.

Solving Equation (24) for the local flow speed V yields

$$V = \sqrt{\frac{2\left(p_0 - p\right)}{\rho}}.$$

Thus if one knows the stagnation pressure and the local pressure one could calculate the corresponding flow speed.

Lecture 8: Bernoulli Equation Continued

September 17, 2025

13.1 The Bernoulli Equation as an Energy Equation

The Bernoulli equation, Equation (22), has been derived by integration of Euler's equation along a streamline for a steady, incompressible, frictionless flow – I.e. it was derived from the momentum equation for a fluid particle.

Another equation, that looks almost identical to Equation (22), although requiring very different restrictions can be obtained from the first law of thermodynamics.

We consider steady flow in the absence of shear forces. We choose a control volume bounded by streamlines as shown on Figure 13.1, which is often called a *stream tube*.

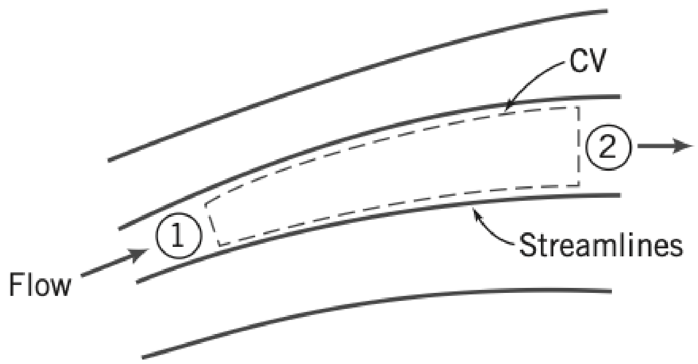


Figure 13.1: Flow through a stream tube.

We have previously defined the first law of thermodynamics on integral form as

$$\dot{Q} - \dot{W}_s - \dot{W}_{\text{shear}} - \dot{W}_{\text{other}} = \frac{\partial}{\partial t}\int_{\text{CV}} e\rho \,\mathrm{d}V + \int_{\text{CS}} \left(e + pv\right)\rho \mathbf{V}\cdot \mathrm{d}\mathbf{A}$$

where

$$e = u + \frac{V^2}{2} + gz.$$

Under the assumptions mentioned above we have

- $\dot{W}_s = 0$,
- $\dot{W}_{\text{shear}} = 0$,
- $\dot{W}_{\text{other}} = 0$,
- steady flow,
- and uniform flow and properties at each section.

This means that the first law of thermodynamics reduces to

$$\left(u_1 + p_1 v_1 + \frac{V_1^2}{2} + gz_1\right)\left(-\rho_1 V_1 A_1\right) + \left(u_2 + p_2 v_2 + \frac{V_2^2}{2} + gz_2\right)\left(\rho_2 V_2 A_2\right) - \dot{Q} = 0.$$

From continuity under the assumptions mentioned above we have:

$$\sum_{\text{CS}} \rho \mathbf{V}\cdot\mathbf{A} = 0$$

or

$$\left(-\rho_1 V_1 A_1\right) + \left((\rho_2 V_2 A_2\right) = 0.$$

I.e.

$$\dot{m} = \rho_1 V_1 A_1 = \rho_2 V_2 A_2.$$

Also

$$\dot{Q} = \frac{\delta Q}{\mathrm{d}t} = \frac{\delta Q}{\mathrm{d}m}\frac{\mathrm{d}m}{\mathrm{d}t} = \frac{\delta Q}{\mathrm{d}m}\dot{m}.$$

Therefore, from the energy equation, after rearranging

$$\left(\left(p_2 v_2 + \frac{V_2^2}{2} + gz_2\right) - \left(p_1 v_1 + \frac{V_1^2}{2} + gz_1\right)\right)\dot{m} + \left(u_2 - u_1 - \frac{\delta Q}{\mathrm{d}m}\right)\dot{m} = 0$$

or

$$p_1 v_1 + \frac{V_1^2}{2} + g z_1 = p_2 v_2 + \frac{V_2^2}{2}.$$

If we also assume the flow is incompressible then $v_1 = v_2 = \frac{1}{\rho}$ and,

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} + g z_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + g z_2 + \left(u_2 - u_1 - \frac{\delta Q}{\mathrm{d}m}\right).$$

We can see that this reduces to the Bernoulli Equation, Equation (22) if the term in parentheses were zero. Therefore if we impose the further restriction,

$$\left(u_2 - u_1 - \frac{\delta Q}{\mathrm{d}m}\right) = 0,$$

the energy equation reduces to

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} + g z_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + g z_2 \implies \frac{p}{\rho} + \frac{V^2}{2} + g z = \text{constant},$$

which is the Bernoulli equation.

Here we note that the Bernoulli equation has been derived in two different ways from entirely different frameworks. It may seem that the restriction, relating heat transfer and internal thermal energy change, is needed for the energy equation to transform to the Bernoulli equation. However, for an incompressible and frictionless flow without shear forces, heat transfer results only in a temperature change and does not affect neither pressure nor velocity.

13.2 Energy Grade Line and Hydraulic Grade Line

For a steady, incompressible, frictionless flow we may use the Bernoulli equation, Equation (22):

$$\frac{p}{\rho} + \frac{V^2}{2} + g z = \text{constant}.$$

If we divide this by g we obtain another form

$$\frac{p}{\rho g} + \frac{V^2}{2g} + z = H.$$

Here H is called the *total head* of the flow and is a measure of the total mechanical energi in units of meters. In a fluid flow H will not be constant but will instead continuously decrease as mechanical energy is converted to thermal. A useful graphical approach is to define this to be the *energy grade line* (EGL),

$$\text{EGL} = \frac{p}{\rho g} + \frac{V^2}{2g} + z.$$

We can also define the *hydraulic grade line* (HGL),

$$\text{HGL} = \frac{p}{\rho g} + z.$$

From these we get

$$\text{EGL} - \text{HGL} = \frac{V^2}{2g},$$

i.e. the difference between the EGL and HGL is the dynamic pressure term. The function of this is shown graphically on Figure 13.2. Here it is worth noting that for incompressible, inviscid flow without work devices the EGL is constant. Furthermore, the HGL will always be lower than the EGL by distance $\frac{V^2}{2g}$.

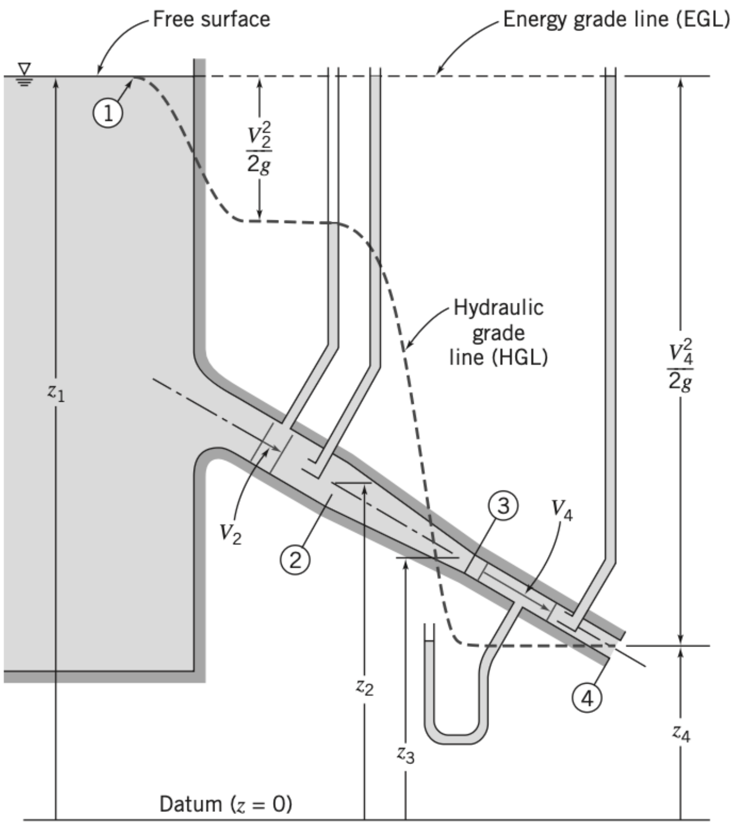


Figure 13.2: Energy and hydraulic grade lines for frictionless flow

13.3 Unsteady Bernoulli Equation: Integration of Euler's Equation Along a Streamline

The Bernoulli equation need not be restricted by the assumption of steady flow. The momentum equation for frictionless flow can be written (with $\mathbf{g}$ in the $-z$ direction) as

$$\frac{\mathrm{D}\mathbf{V}}{\mathrm{D}t} = -\frac{1}{\rho}\nabla p - g\hat{k}.$$

This is a vector equation, which can be converted to a scalar equation by taking the dot product with $\mathrm{d}\mathbf{s}$, where $\mathrm{d}\mathbf{s}$ is an element of distance along a streamline. Thus

$$\frac{\mathrm{D}\mathbf{V}}{\mathrm{D}t}\cdot \mathrm{d}\mathbf{s} = \frac{\mathrm{D}V}{\mathrm{D}t}\,\mathrm{d}s = V\frac{\partial V}{\partial s}\,\mathrm{d}s + \frac{\partial V}{\partial t}\,\mathrm{d}s = -\frac{1}{\rho}\nabla p\cdot \mathrm{d}\mathbf{s} - g\hat{k}\cdot \mathrm{d}\mathbf{s}.$$

Here we note that

$$\frac{\partial V}{\partial s}\,\mathrm{d}s = \mathrm{d}V \qquad \text{(the change in } V \text{ along } s\text{)}$$

$$\nabla p\cdot \mathrm{d}\mathbf{s} = \mathrm{d}p \qquad \text{(the change in pressure along } s\text{)}$$

$$\hat{j}\cdot \mathrm{d}\mathbf{s} = \mathrm{d}z \qquad \text{(the change in } z \text{ along } s\text{).}$$

Substituting these into the expression we obtain

$$V\,\mathrm{d}V + \frac{\partial V}{\partial t}\,\mathrm{d}s = -\frac{\mathrm{d}p}{\rho} - g\,\mathrm{d}z.$$

Integrating this along a streamline from a point, 1, to another point, 2, yields

$$\int_1^2 \frac{\mathrm{d}p}{\rho} + \frac{V_2^2 - V_1^2}{2} + g\left(z_2 - z_1\right) + \int_1^2 \frac{\partial V}{\partial t}\,\mathrm{d}s = 0.$$

For incompressible flow, ρ = constant, and the above equation then reduces to

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + gz_2 + \int_1^2 \frac{\partial V}{\partial t}\,\mathrm{d}s.$$

This is applicable for incompressible, frictionless flow along a streamline. This is also known as the Bernoulli equation for unsteady flow. It differs from the Bernoulli equation by the term $\int_1^2 \frac{\partial V}{\partial t}\,\mathrm{d}s$. This term can be interpreted as the work involved in the rate of increase of momentum of the fluid on the streamline over time, as opposed to the change in momentum over distance, represented by the change in velocity from V_1 to V_2.

13.4 Irrotational Flow

Irrotational flows were briefly mentioned in Section 10.1.2. These are flows in which the fluid particles do not rotate ($\boldsymbol{\omega} = 0$). The only stresses that can generate particle rotation are shear stresses. Therefore, all inviscid flows are inherently irrotational unless the particles were somehow initially rotating. The irrotationality condition is

$$\nabla \times \mathbf{V} = 0 \implies \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0.$$

13.4.1 Bernoulli Equation Applied to Irrotational Flow

The Bernoulli equation has been shown to hold along a streamline for a steady, incompressible, inviscid flow. If in addition to this, the flow field is also irrotational (i.e. the particles had no initial rotation), such that $\nabla \times \mathbf{V} = 0$, then Bernoulli's equation will be the same value for all streamlines – this means that for irrotational flows the Bernoulli equation is also valid across different streamlines.

To show this we start with Euler's equation in vector form,

$$(\mathbf{V} \cdot \nabla)\,\mathbf{V} = -\frac{1}{\rho}\nabla p - g\hat{k}.$$

Using the vector identity

$$(\mathbf{V} \cdot \nabla)\,\mathbf{V} = \frac{1}{2}\nabla\,(\mathbf{V} \cdot \mathbf{V}) - \mathbf{V} \times (\nabla \times \mathbf{V}),$$

we see that for irrotational flow, when $\nabla \times \mathbf{V} = 0$,

$$(\mathbf{V} \cdot \nabla)\,\mathbf{V} = \frac{1}{2}\nabla\,(\mathbf{V} \cdot \mathbf{V})$$

hence Euler's equation for irrotational flow can be written as

$$\frac{1}{2}\nabla\,(\mathbf{V} \cdot \mathbf{V}) = \frac{1}{2}\nabla\left(V^2\right) = -\frac{1}{\rho}\nabla p - g\hat{k}.$$

We consider an arbitrary displacement from $\mathbf{r}$ to $\mathbf{r} + \mathrm{d}\mathbf{r}$ not necessarily along a streamline. Taking the dot product of this with each term in the above yields

$$\frac{1}{2}\nabla\left(V^2\right) \cdot \mathrm{d}\mathbf{r} = -\frac{1}{\rho}\nabla p \cdot \mathrm{d}\mathbf{r} - g\hat{k} \cdot \mathrm{d}\mathbf{r}$$

hence

$$\frac{1}{2}\mathrm{d}\left(V^2\right) = -\frac{\mathrm{d}p}{\rho} - g\,\mathrm{d}z$$

or

$$\frac{\mathrm{d}p}{\rho} + \frac{1}{2}\mathrm{d}\left(V^2\right) + g\,\mathrm{d}z = 0.$$

Integrating this for an incompressible flow gives

$$\frac{p}{\rho} + \frac{V^2}{2} + gz = \text{constant}$$

which is the Bernoulli equation. Therefore it has been shown that this holds between any two general points in an irrotational flow.

Lecture 9: Dimensional Analysis September 22, 2025

15 Dimensional Analysis and Similitude

Many phenomena in fluid mechanics depend on complex interplay between geometry and flow phenomena. E.g. consider the drag force on a stationary smooth sphere immersed in a uniform stream. We expect this to depend on the size of the sphere (defined by the diameter, D), the fluid speed V, the fluid density ρ and the fluid viscosity μ. If the drag force is called F this can be written symbolically as

$$F = f(D, V, \rho, \mu) \tag{25}$$

If we then wanted to determine how F depends on V, D, ρ, and μ we could not simply vary one parameter at a time in an experimental setting one by one as with just four parameters with 10 different experimental runs at different values each we have 10^4 different experiments.

Instead of doing this extremely cumbersome task we can use dimensional analysis to show that all the data for drag on a smooth sphere can be plotted as a single relationship between two nondimensional parameters in the form

$$\frac{F}{\rho V^2 D^2} = f\left(\frac{\rho V D}{\mu}\right).$$

The form of the function f must still be determined experimentally, however all spheres, in all fluids, for most velocities are expected to fall on this same function curve. This includes everything from the drag on a hot-air ballon to red blood cells moving through an aorta.

15.1 Nondimensionalizing the Basic Differential Equations

We consider a steady, incompressible two-dimensional flow of a Newtonian fluid with constant viscosity. The mass conservation equation becomes

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{26}$$

and the Navier-Stokes equations reduce to

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{27}$$

$$\rho\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = -\rho g - \frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right). \tag{28}$$

These equations form a set of coupled nonlinear partial differential equations for u, v, and p, and are rather cumbersome to solve for most flows. The mass conservation equation, Equation (26), here has dimensions of $\left[\frac{1}{\text{time}}\right]$, and the Navier-Stokes equations, Equations (27) and (28), have dimensions of $\left[\frac{\text{force}}{\text{volume}}\right]$. We will now try to convert these to dimensionless equations.

To nondimensionalize these equations, we divide all lengths be a reference length L and all the velocities by a reference speed V_∞ often taken as the freestream velocity. Furthermore, we make the pressure nondimensional by dividing by ρV_∞^2. If we denote the nondimensional quantities with asterisks, we obtain

$$x^* = \frac{x}{L}, \qquad y^* = \frac{y}{L}, \qquad u^* = \frac{u}{V_\infty}, \qquad v^* \frac{v}{V_\infty}, \qquad p^* = \frac{p}{\rho V_\infty^2}.$$

If we substitute this into Equations (26) to (28) we get

$$\frac{V_\infty}{L}\frac{\partial u^*}{\partial x^*} + \frac{V_\infty}{L}\frac{\partial v^*}{\partial y^*} = 0 \tag{29}$$

$$\frac{\rho V_\infty^2}{L}\left(u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*}\right) = -\frac{\rho V_\infty^2}{L}\frac{\partial p^*}{\partial x^*} + \frac{\mu V_\infty}{L^2}\left(\frac{\partial^2 u^*}{\partial x^{*2}} + \frac{\partial^2 u^*}{\partial y^{*2}}\right) \tag{30}$$

$$\frac{\rho V_\infty^2}{L}\left(u^*\frac{\partial v^*}{\partial x^*} + v^*\frac{\partial v^*}{\partial y^*}\right) = -\rho g - \frac{\rho V_\infty^2}{L}\frac{\partial p^*}{\partial y^*} + \frac{\mu V_\infty}{L^2}\left(\frac{\partial^2 u^*}{\partial x^{*2}} + \frac{\partial^2 v^*}{\partial y^{*2}}\right). \tag{31}$$

Dividing Equation (29) by $\frac{V_\infty}{L}$ and Equations (30) and (31) by $\frac{\rho V_\infty^2}{L}$ yields

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0 \tag{32}$$

$$u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial y^*}{\partial y^*} = -\frac{\partial p^*}{\partial x^*} + \frac{\mu}{\rho V_\infty L}\left(\frac{\partial^2 u^*}{\partial x^{*2}} + \frac{\partial^2 u^*}{\partial x^{*2}} + \frac{\partial^2 u^*}{\partial y^{*1}}\right) \tag{33}$$

$$u^*\frac{\partial v^*}{\partial x^*} + v^*\frac{\partial v^*}{\partial y^*} = -\frac{gL}{V_\infty^2} - \frac{\partial p^*}{\partial y^*} + \frac{\mu}{\rho V_\infty L}\left(\frac{\partial^2 v^*}{\partial x^{*2}} + \frac{\partial^2 v^*}{\partial y^{*2}}\right). \tag{34}$$

Equations (32) to (34) are the nondimensonal forms of our original equations, Equations (26) to (28).

15.2 Buckingham Pi Theorem

At the start of Section 15 it was discussed how the drag F on a sphere can be modelled as

$$F = F(D, \rho, \mu, V)$$

where either theory or experiment is needed to determine the nature of the function f. More formally, we write

$$g(F, D, \rho, \mu, V) = 0$$

where g is an unspecified function, different from f. The Buckingham Pi theorem states that we can transform a relationship between n parameters of the form

$$g(q_1, q_2, \ldots, q_n) = 0$$

into a corresponding relationship between $n - m$ independent dimensionless Π parameters in the form

$$G(\Pi_1, \Pi_2, \ldots, \Pi_{n-m}) = 0$$

or

$$\Pi_1 = G_1(\Pi_2, \ldots, \Pi_{n-m})$$

where m usually is the minimum number r of independent dimensions (e.g. mass, length, time) required to define the dimensions of all the parameters $q_1, q_2, \ldots q_n$. For the sphere problem it can be shown that

$$g(F, D, \rho, \mu, V) = 0 \qquad \text{or} \qquad F = F(D, \rho, \mu, V)$$

leads to

$$G\left(\frac{F}{\rho V^2 D^2}, \frac{\mu}{\rho V D}\right) = 0 \qquad \text{or} \qquad \frac{F}{\rho V^2 D^2} = G_1\left(\frac{\mu}{\rho V D}\right).$$

The theorem does not predict the form of neither G nor G_1. I.e. the functional relation among the independent dimensionless Π parameters must be determined experimentally.

The $n-m$ dimensionless Π parameters obtained from this procedure are independent. A Π parameter is not said to be independent if it can be formed from any combination of one or more of the other Π parameters.

The following six steps outline the normal procedure for determining the Π parameters

1. List all the dimensional parameters involved. Let n be the number of parameters. If not all the important parameters are included then a relation will still be obtained however it will not give the full story. If parameters that actually have no effect are included these will either vanish during the dimensional analysis or result in extraneous dimensional groups that do not have any effect.
2. Select a set of fundamental (or primary) dimensions, e.g. MLt or FLt. For a heat transfer problem it might be wise to include T for temperature and in electrical systems q for charge and so on.
3. List the dimensions of all parameters in terms of primary dimensions. Let r be the number of primary dimensions. Either force or mass may be selected as a primary dimension.
4. Select a set of r dimensional parameters that includes all the primary dimensions. These parameters will all be combined with each of the remaining parameters, one at a time, and will therefore be called *repeating parameters*. No repeating parameter should have dimensions that are a power of the dimensions of another repeating parameter, e.g. do not include both an area $[L^2]$ and a second moment $[L^4]$ as repeating parameters. The repeating parameter chosen may appear in all the dimensionless groups obtained. Therefore one should not include the dependent parameter in the repeating parameters.
5. Set up dimensional equations, combining the parameters selected in the previous step with each of the other parameters, in turn, to form dimensionless groups. There will be $n-m$ equations. Solve the dimensional equations to obtain the $n-m$ dimensionless groups.
6. Check to see that each group obtained is actually dimensionless. If mass was initially selected as a primary dimension, it is wise to check the groups using force as a primary dimension, or vice versa.

The functional relationship among the Π parameters obtained from the above method must be determined by experiment.

The $n-m$ dimensionless groups obtained by this procedure are independent but not unique. If a different set of repeating parameters is chosen different dimensionless groups are found. Based on experience viscosity should appear in only one dimensionless parameter. Therefore μ should not be chosen as a repeating parameter as these may appear in all the dimensionless groups obtained.

It usually works best to choose density $\rho = \left[\frac{M}{L^3}\right]$, speed $V = \left[\frac{L}{t}\right]$, and characteristic length L as repeating parameters as these typically lead to dimensionless parameters suitable for a wide range of data and they are all easily measurable.

15.3 Significant Dimensionless Groups in Fluid Mechanics

Many important dimensionless groups have been identified. Several of these are so fundamental they are worth remembering.

The typical forces encountered in flowing fluids are due to inertia, viscosity, pressure, gravity, surface tension, and compressibility. The ratio of any two forces will be dimensionless. It has previously been shown that the inertia force is proportional to $\rho V^2 L^2$. hence, we can compare the relative magnitude of various fluid forces to the inertia force:

$$
\begin{array}{llclcll}
\text{Viscous Force} & \sim & \tau A = \mu \dfrac{\mathrm{d}u}{\mathrm{d}y} A \propto \mu \dfrac{V}{L} L^2 = \mu V L & \text{so} & \dfrac{\text{viscous}}{\text{inertia}} & \sim & \dfrac{\mu V L}{\rho V^2 L^2} = \dfrac{\mu}{\rho V L} \\
\text{Pressure Force} & \sim & \Delta p A \propto \Delta p L^2 & \text{so} & \dfrac{\text{pressure}}{\text{inertia}} & \sim & \dfrac{\Delta p L^2}{\rho V^2 L^2} = \dfrac{\Delta p}{\rho V^2} \\
\text{Gravity Force} & \sim & mg \propto g \rho L^3 & \text{so} & \dfrac{\text{gravity}}{\text{inertia}} & \sim & \dfrac{g \rho L^3}{\rho V^2 L^2} = \dfrac{gL}{V^2} \\
\text{Surface Tension} & \sim & \sigma L & \text{so} & \dfrac{\text{Surface Tension}}{\text{inertia}} & \sim & \dfrac{\sigma L}{\rho V^2 L^2} = \dfrac{\sigma}{\rho V^2 L} \\
\text{Compressibility Force} & \sim & E_v A \propto E_v L^2 & \text{so} & \dfrac{\text{Compressibility}}{\text{inertia}} & \sim & \dfrac{E_v L^2}{\rho V^2 L^2} = \dfrac{E_v}{\rho V^2}.
\end{array}
$$

All of these occur so frequently that they have been given identifying names.

The first parameter $\frac{\mu}{\rho V L}$, is by tradition inverted to the form $\frac{\rho V L}{\mu}$. This is termed *Reynolds number* after Osborne Reynolds who discovered that

$$\mathrm{Re} = \frac{\rho \overline{V} D}{\mu} = \frac{\overline{V} D}{\nu}$$

is a criterion by which the flow regime may be determined. In general this is

$$\mathrm{Re} = \frac{\rho V L}{\mu} = \frac{V L}{\nu}$$

where L is a characteristic length descriptive of the flow field geometry. The Reynolds number can be seen as the ratio of inertia forces to viscous forces. Flows with “large” Reynolds numbers are generally more turbulent and flows in which the inertia forces are “small” compared to the viscous forves are characteristically laminar.

In aerodynamics, the second parameter, $\frac{\Delta p}{\rho V^2}$ is normally modified by inserting a factor $\frac{1}{2}$ in the denominator such that it represents the dynamic pressure. The ratio

$$\mathrm{Eu} = \frac{\Delta p}{\frac{1}{2} \rho V^2}$$

is formed, where Δp is the local pressure minus the freestream pressure, and ρ and V are properties of the freestream flow. This ratio is called the *Euler number* (not to be confused with Euler's number e) named after Leonhard Euler. This is also sometimes called the *pressure coefficient*, C_p.

When studying cavitation, the pressure difference, Δp, is taken as $\Delta p = p - p_v$ where p is the pressure in the liquid stream and p_v is the vapor pressure at the test temperature. Combining these with ρ and V in the stream yields the dimensionless parameter called the *cavitation number*,

$$\mathrm{Ca} = \frac{p - p_v}{\frac{1}{2} \rho V^2}.$$

The smaller the cavitation number, the more likely cavitation is to occur. This is usually unwanted.

William Froude and his son Robert E. Froude are credited with discovering that the parameter

$$\mathrm{Fr} = \frac{V}{\sqrt{gL}}$$

was significant for flows with free surface effects. Squaring the *Froude number* gives

$$\mathrm{Fr}^2 = \frac{V^2}{gL}$$

which is the ratio of inertia to gravity (the inverse of the third ratio above). The length L is, once again, a characteristic length descriptive of the flow field. For open-channel flow L will equal the water depth. Froude numbers less than unity indicate subcritical flow and values greater than unity represents supercritical flow.

By convention the inverse of the fourth force ratio $\frac{\sigma}{\rho V^2 L}$ is called the *Weber number* and indicates the ratio of inertia to surface tension forces,

$$\mathrm{We} = \frac{\rho V^2 L}{\sigma}.$$

The value of the Weber number is indicative of the existence of and frequency of capillary waves at a free surface.

Ernst Mach introduced the parameter

$$M = \frac{V}{c}$$

where V is the flow speed and c is the local sonic speed. This is a key parameter that characterizes compressibility effects in a flow. This may also be written as:

$$M = \frac{V}{c} = \frac{V}{\sqrt{\frac{\mathrm{d}p}{\mathrm{d}\rho}}} = \frac{V}{\sqrt{\frac{E_v}{\rho}}} \implies M^2 = \frac{\rho V^2 L^2}{E_v L^2} = \frac{\rho V^2}{E_v}$$

which is the inverse of the final force ratio $\frac{E_v}{\rho V^2}$ discussed above. For truly incompressible flow, $c = \infty$ so $M = 0$.

Lecture 10: Internal Flows: Fully Developed Laminar Flows September 24, 2025

17 Internal Incompressible Viscous Flow

17.1 Internal Flow Characteristics

17.1.1 Laminar Versus Turbulent Flow

The pipe flow regime, i.e. if the flow is turbulent or laminar, is determined by the Reynolds number, briefly mentioned in Section 15.3, $\mathrm{Re} = \frac{\rho \overline{V} D}{\mu}$. Under normal conditions, the transition to turbulence occurs at roughly $\mathrm{Re} \approx 2300$ for flow in pipes. For water in a $\varnothing = 1\,\mathrm{in}$ pipe, this corresponds to an average speed of $V \approx 0{,}1\,\frac{\mathrm{m}}{\mathrm{s}}$. Turbulence occurs when the viscous forces are not able to damp out the random fluctuations in fluid motion. This means that a fluid of higher viscosity (e.g. motor oil compared to water) is able to be flowing at a higher rate before becoming turbulent. On the other hand, high-density fluids will exacerbate the inertial forces due to random fluctuations in motion, and therefore this fluid will become turbulent at a lower flow rate.

17.1.2 The Entrance Region

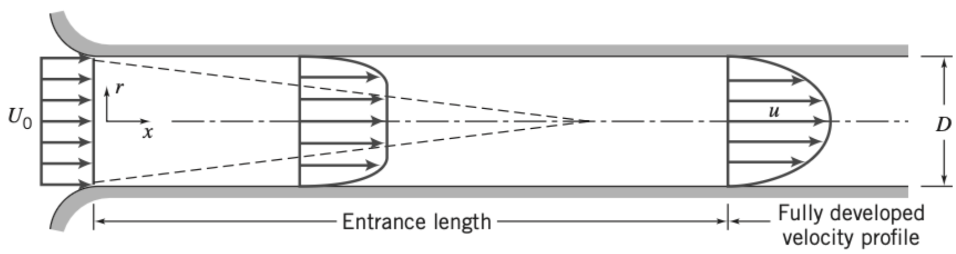


Figure 17.1: Flow in the entrance region of a pipe.

On Figure 17.1 laminar flow in the entrance region of a circular pipe is portrayed. The flow has uniform velocity U_0 at the pipe entrance. Due to the no-slip condition, we know that the velocity along the wall must be zero along the entire length of the pipe. Therefore, a boundary layer will develop along the wall. As the fluid flows through the pipe the shear force from the pipe walls will slowly lead to this boundary layer expanding until the fully developed velocity profile is reached. For incompressible flow, mass conservation requires that as the speed close to the wall is reduced, the speed in the central frictionless region of the pipe must increase to compensate.

A *fully developed flow* is one which does not change velocity profile with distance. The distance from the entrance to the point at which the flow is fully developed is termed the *entrance length*.

For laminar flow the entrance length L, is a function of Reynolds number,

$$\frac{L}{D} \approx 0{,}06\frac{\rho \overline{V} D}{\mu}$$

where $\overline{V} = \frac{Q}{A}$ is the average velocity. If the flow is instead turbulent, the enhanced mixing of fluid layers will lead to the boundary layer growing quicker and therefore lead to a shorter entrance length.

17.2 Fully Developed Laminar Flow Between Infinite Parallel Plates

17.2.1 Both Plates Stationary

Fluid in high-pressure hydraulic systems, such as automobile brake systems, often leaks through the annular gap between a piston and cylinder, as shown on Figure 17.2. For very small gaps this flow field may be modeled as flow between infinite parallel plates. To calculate the leakage flow rate, we will first determine the velocity field.

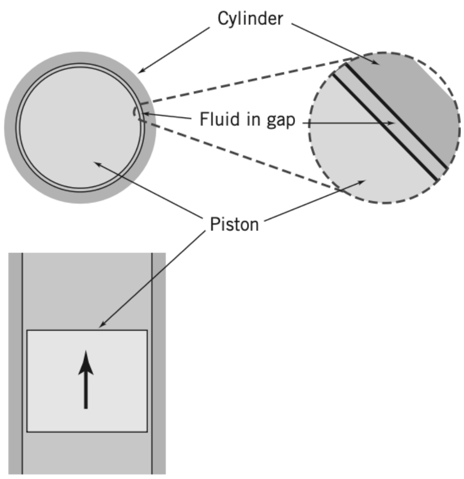


Figure 17.2: Piston-cylinder system approximated as infinite parallel plates.

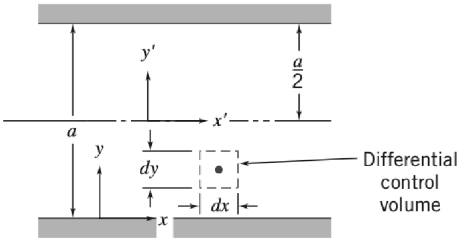
We consider the fully developed laminar flow between horizontal infinite plates. The plates are separated by a distance a and considered infinite in the z-direction with no variation of any fluid property in this direction. The flow is also assumes steady and incompressible. From the no-slip condition the boundary conditions are

$$\begin{aligned} y &= 0 && \implies u = 0 \\ y &= a && \implies u = 0. \end{aligned}$$

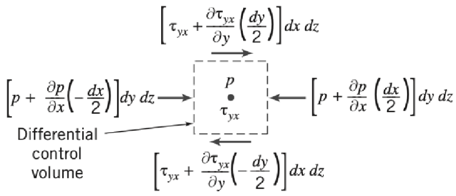
Since the flow is fully developed it cannot vary with x and, hence, depends only on y, so that $u = u(y)$. Furthermore, there will be no component of velocity in the y or z directions, as the flow is fully developed.

We consider a differential control volume of side $\mathrm{d}V = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ as shown on fig. 17.3. To this control volume we apply the momentum equation in the x direction.

$$F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{\mathrm{CV}} u\rho\,\mathrm{d}V + \int_{\mathrm{CS}} u\rho \boldsymbol{V} \cdot \mathrm{d}\boldsymbol{A}.$$



(a) Geometry of CV



(b) Forces acting on CV

Figure 17.3: Control volume for analysis of laminar flow between stationary infinite parallel plates.

In this $F_{B_x} = 0$ due to the assumed geometry and as the flow is assumed steady the first term on the RHS will also equal zero. Hence,

$$F_{S_x} = \int_{\mathrm{CS}} u\rho \boldsymbol{V}\cdot \mathrm{d}\boldsymbol{A}.$$

For fully developed flow, a basic feature is that the velocity profile is the same at all locations along the flow; hence there is no change in momentum. Therefore the above equation reduces to the simple result,

$$F_{S_x} = 0.$$

Therefore the next step is to sum up the forces acting on the control volume in the x direction. These are the pressure forces on the left and right faces and the shear forces on the top and bottom faces.

If we assume the pressure at the center of the element to be p, then the pressure force on the left face is

$$\mathrm{d}F_L = \left(p - \frac{\mathrm{d}p}{\mathrm{d}x}\frac{\mathrm{d}x}{2}\right)\mathrm{d}y\,\mathrm{d}z$$

and the pressure force on the right face is

$$\mathrm{d}F_R = -\left(p + \frac{\mathrm{d}p}{\mathrm{d}x}\frac{\mathrm{d}x}{2}\right)\mathrm{d}y\,\mathrm{d}z.$$

Likewise, if the shear stress at the center of the element is assumed to be τ_{yx}, then the shear force on the bottom face is

$$\mathrm{d}F_B = -\left(\tau_{yx} - \frac{\mathrm{d}\tau_{yx}}{\mathrm{d}y}\frac{\mathrm{d}y}{2}\right)\mathrm{d}x\,\mathrm{d}z$$

and the shear force on the top face is

$$\mathrm{d}F_T = \left(\tau_{yx} + \frac{\mathrm{d}\tau_{yx}}{\mathrm{d}y}\frac{\mathrm{d}y}{2}\right)\mathrm{d}x\,\mathrm{d}z.$$

Inserting all of these forces in to the momentum equation from before yields

$$\frac{\mathrm{d}p}{\mathrm{d}x} = \frac{\mathrm{d}\tau_{yx}}{\mathrm{d}y}.$$

This equation states that as the momentum is constant, the pressure force balances the friction force. Here the LHS is only a function of x and the RHS is only a function of y. Hence, the only way for it to be valid for all x and y is if each side is in fact constant:

$$\frac{\mathrm{d}p}{\mathrm{d}x} = \frac{\mathrm{d}\tau_{yx}}{\mathrm{d}y} = \text{constant}.$$

Upon integration of this we obtain

$$\tau_{yx} = \left(\frac{\mathrm{d}p}{\mathrm{d}x}\right) y + c_1$$

which indicates that the shear stress varies linearly with y. For a Newtonian fluid in one dimension the shear stress is also given by

$$\tau_{yx} = \mu \frac{\mathrm{d}u}{\mathrm{d}y}$$

Upon equating these and integrating for y the following expression is reached

$$u = \frac{1}{2\mu}\left(\frac{\mathrm{d}p}{\mathrm{d}x}\right) y^2 - \frac{1}{2\mu}\left(\frac{\mathrm{d}p}{\mathrm{d}x}\right) a y = \frac{a^2}{2\mu}\left(\frac{\mathrm{d}p}{\mathrm{d}x}\right)\left(\left(\frac{y}{a}\right)^2 - \left(\frac{y}{a}\right)\right).$$

This is the velocity profile of the flow.

The shear stress distribution is then

$$\tau_{yx} = \left(\frac{\mathrm{d}p}{\mathrm{d}x}\right) y + c_1 = \left(\frac{\mathrm{d}p}{\mathrm{d}x}\right) y - \frac{1}{2}\left(\frac{\mathrm{d}p}{\mathrm{d}x}\right) a = a\left(\frac{\mathrm{d}p}{\mathrm{d}x}\right)\left(\frac{y}{a} - \frac{1}{2}\right).$$

The volume flow rate per unit depth is

$$\frac{Q}{l} = -\frac{1}{12\mu}\left(\frac{-\Delta p}{L}\right) a^3 = \frac{a^3 \Delta p}{12\mu L}.$$

The average velocity $\overline{V}$ is given by

$$\overline{V} = \frac{Q}{A} = -\frac{1}{12\mu}\left(\frac{\mathrm{d}p}{\mathrm{d}x}\right)\frac{a^3 l}{la} = -\frac{1}{12\mu}\left(\frac{\mathrm{d}p}{\mathrm{d}x}\right) a^2.$$

The point of maximum velocity will be at $y = \frac{a}{2}$ where the velocity will be

$$u_{y=\frac{a}{2}} = u_{\max} = -\frac{1}{8\mu}\left(\frac{\mathrm{d}p}{\mathrm{d}x}\right) a^2 = \frac{3}{2}\overline{V}.$$

17.2.2 Upper Plate Moving with Constant Speed

The second basic way to generate flow between infinite parallel plates, is to have one plate moving parallel to the either.

In this case the boundary conditions are

$$\begin{aligned} u &= 0 && \implies && y = 0 \\ u &= U && \implies && y = a. \end{aligned}$$

In this case the velocity distribution is also given by

$$u = \frac{1}{2\mu}\left(\frac{\mathrm{d}p}{\mathrm{d}x}\right) y^2 + \frac{c_1}{\mu} y + c_2.$$

Evaluating for c_1 and c_2 using the boundary conditions yields

$$u = \frac{Uy}{a} + \frac{a^2}{2\mu}\left(\frac{\mathrm{d}p}{\mathrm{d}x}\right)\left(\left(\frac{y}{a}\right)^2 - \left(\frac{y}{a}\right)\right).$$

The shear stress distribution will be

$$\tau_{yx} = \mu\frac{U}{a} + \frac{a^2}{2}\left(\frac{\mathrm{d}p}{\mathrm{d}x}\right)\left(\frac{2y}{a^2} - \frac{1}{a}\right) = \mu\frac{U}{a} + a\left(\frac{\mathrm{d}p}{\mathrm{d}x}\right)\left(\frac{y}{a} - \frac{1}{2}\right).$$

The volume flow rate per unit depth is

$$\frac{Q}{l} = \frac{Ua}{2} - \frac{1}{12\mu}\left(\frac{\mathrm{d}p}{\mathrm{d}x}\right)a^3.$$

The average velocity magnitude $\overline{V}$ is

$$\overline{V} = \frac{Q}{A} = \frac{l\left(\frac{Ua}{2} - \frac{1}{12\mu}\left(\frac{\mathrm{d}p}{\mathrm{d}x}\right)a^3\right)}{la} = \frac{U}{2} - \frac{1}{12\mu}\left(\frac{\mathrm{d}p}{\mathrm{d}x}\right)a^2.$$

The point of maximum velocity will be at

$$y = \frac{a}{2} - \frac{\frac{U}{a}}{\left(\frac{1}{\mu}\right)\left(\frac{\mathrm{d}p}{\mathrm{d}x}\right)}.$$

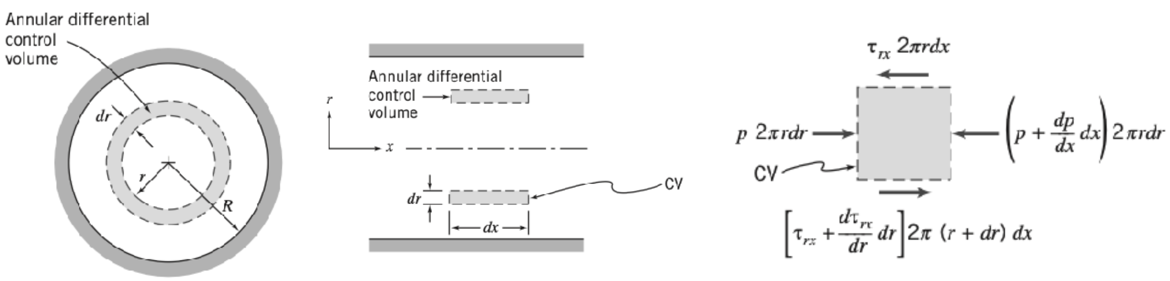
However there is no simple relationship for relating the max flow velocity at this point to the mean velocity in this case.

17.3 Fully Developed Laminar Flow in a Pipe

Flow in a pipe is axisymmetric. Consequently it is most convenient to work in cylindrical coordinates when describing these.

For a fully developed steady flow, the x component of the momentum equation has been shown above to reduce to

$$F_{S_x} = 0.$$



(a) End view of CV (b) Side view CV (c) Forces on CV

Figure 17.4: Differential control volume for analysis of fully developed laminar flow in a pipe.

We consider the differential control volume shown on Figure 17.4. Using the same notion as in the previous section the pressure force on the left face of the control volume is

$$\mathrm{d}F_L = p2\pi r\,\mathrm{d}r$$

and the pressure force on the right end is

$$\mathrm{d}F_R = -\left(p + \frac{\mathrm{d}p}{\mathrm{d}x}\mathrm{d}x\right)2\pi r\,\mathrm{d}r.$$

The shear force on the inner cylindrical surface of the control volume is

$$\mathrm{d}F_I = -\tau_{rx} 2\pi r \, \mathrm{d}x$$

and the shear force on the outer cylindrical surface is

$$\mathrm{d}F_O = \left(\tau_{rx} + \frac{\mathrm{d}\tau_{rx}}{\mathrm{d}r}\,\mathrm{d}r\right) 2\pi\,(r + \mathrm{d}r)\,\mathrm{d}x.$$

Substituting all of these into the momentum equation from before leads to

$$-\frac{\mathrm{d}p}{\mathrm{d}x} 2\pi r \,\mathrm{d}r\,\mathrm{d}x + \tau_{rx} 2\pi \,\mathrm{d}r\,\mathrm{d}x + \frac{\mathrm{d}\tau_{rx}}{\mathrm{d}r} 2\pi r\,\mathrm{d}r\,\mathrm{d}x = 0.$$

Dividing this by $2\pi r$ and solving for $\frac{\partial p}{\partial x}$ gives

$$\frac{\partial p}{\partial x} = \frac{\tau_{rx}}{r} + \frac{\mathrm{d}\tau_{rx}}{\mathrm{d}r} = \frac{1}{r}\frac{\mathrm{d}\,(r\tau_{rx})}{\mathrm{d}r}.$$

Here the same argument as we used for the parallel plates of the LHS only depending on x and the RHS only depending on r meaning both must be constant:

$$\frac{1}{r}\frac{\mathrm{d}\,(r\tau_{rx})}{\mathrm{d}r} = \frac{\mathrm{d}p}{\mathrm{d}x} = \text{constant} \qquad \text{or} \qquad \frac{\mathrm{d}\,(r\tau_{rx})}{\mathrm{d}r} = r\frac{\mathrm{d}p}{\mathrm{d}x}.$$

This shows that in a constant diameter pipe, the pressure drops uniformly along the pipe length. Integrating this and employing the boundary conditions yields

$$u = -\frac{R^2}{4\mu}\left(\frac{\mathrm{d}p}{\mathrm{d}x}\right)\left(1 - \left(\frac{r}{R}\right)^2\right).$$

The shear stress distribution is

$$\tau_{rx} = \mu\frac{\mathrm{d}u}{\mathrm{d}r} = \frac{r}{2}\left(\frac{\mathrm{d}p}{\mathrm{d}x}\right).$$

The volume flow rate is

$$Q = -\frac{\pi R^4}{8\mu}\left(\frac{\mathrm{d}p}{\mathrm{d}x}\right) = \frac{\pi \Delta p R^4}{8\mu L}.$$

The average velocity is

$$\overline{V} = \frac{Q}{A} = \frac{Q}{\pi R^2} = -\frac{R^2}{8\mu}\left(\frac{\mathrm{d}p}{\mathrm{d}x}\right).$$

The point of maximum velocity will be directly at the center of the pipe where the velocity will be

$$u_{r=0} = u_{\max} = -\frac{R^2}{4\mu}\left(\frac{\mathrm{d}p}{\mathrm{d}x}\right) = 2\overline{V}.$$

Lecture 11: Internal Flows: Flows in Pipes and Ducts

September 29, 2025

18.1 Shear Stress Distribution in Fully Developed Pipe Flow

We, once again, consider fully developed flow in a horizontal circular pipe, except this time the flow is not restricted to being laminar. Above it was shown that a force balance between friction and pressure forces leads to

$$\tau_{rx} = \frac{r}{2}\left(\frac{\mathrm{d}p}{\mathrm{d}x}\right) + \frac{c_1}{r}.$$

As we cannot have infinite stress at the centerline, c_1 must be zero, so

$$\tau_{rx} = \frac{r}{2}\frac{\mathrm{d}p}{\mathrm{d}x}.$$

This indicates that for both laminar and turbulent fully developed flows the shear stress varies linearly across the pipe from zero at the centerline to a maximum at the pipe wall.

Turbulent flow is represented at each point in time by a mean velocity $\overline{u}$ plus randomly fluctuating velocity components u' and v' in the x and y directions respectively. These components continuously transfer momentum between adjacent fluid layers, tending to reduce any velocity gradient present. This results in an apparent stress termed the *Reynolds stress* given by $-\rho\overline{u'v'}$. In terms of distance from the wall, the total stress in turbulent flow can be written as

$$\tau = \tau_{\text{lam}} + \tau_{\text{turb}} = \mu\frac{\mathrm{d}\overline{u}}{\mathrm{d}y} - \rho\overline{u'v'} \tag{35}$$

In the above, the fluctuations u' and v' are negatively correlated, so that $\tau_{\text{turb}} = -\rho\overline{u'v'}$ is positive. On Figure 18.1 experimental measurements of the Reynolds stress for two fully developed turbulent pipe flows are plotted.

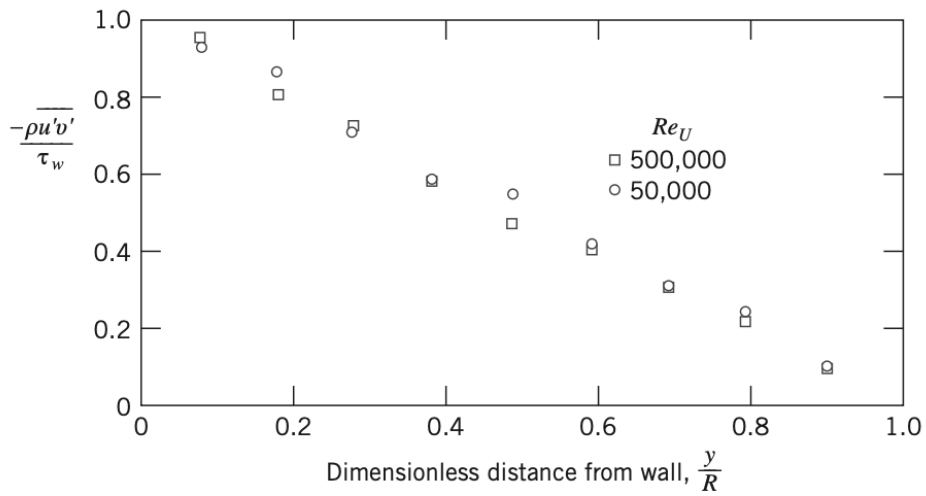


Figure 18.1: Turbulent shear stress (Reynold's stress) for fully developed turbulent flow in a pipe.

18.2 Turbulent Velocity Profiles in Fully Developed Pipe Flow

We consider the relation for the total stress in turbulent flow from Equation (35) and divide by the density to get

$$\frac{\tau}{\rho} = \nu\frac{\mathrm{d}\overline{u}}{\mathrm{d}y} - \overline{u'v'}.$$

It is convenient to consider the turbulent fluctuations in the mean velocity as a result of a small element of fluid moving upward a small distance $\mathscr{L}$ into a region of higher velocity. The fluctuating velocity u' is then negative and given by

$$u' = -\mathscr{L}\frac{\mathrm{d}\overline{u}}{\mathrm{d}y}$$

where $\mathscr{L}$ is the so-called mixing length. Similarly for the v component of velocity

$$v' = \mathscr{L}\frac{\mathrm{d}\overline{v}}{\mathrm{d}x}.$$

In homogeneous turbulence u' and v' are equal and therefore

$$v' = u' = \mathscr{L}\frac{\mathrm{d}\overline{u}}{\mathrm{d}y}.$$

The turbulent shear stress is then approximated by

$$\frac{\tau_{\text{turb}}}{\rho} = \mathscr{L}^2\left(\frac{\mathrm{d}\overline{u}}{\mathrm{d}y}\right)^2.$$

Furthermore, the mixing length approximately increases in proportion to the distance from the wall,

$$\mathscr{L} = ky.$$

In general, the velocity profile for turbulent flow through a smooth pipe may be approximated by the power-law equation

$$\frac{\overline{u}}{U} = \left(\frac{y}{R}\right)^{\frac{1}{n}} = \left(1-\frac{r}{R}\right)^{\frac{1}{n}}$$

where the exponent n varied with the Reynold's number and is approximately

$$n = -1{,}7 + 1{,}8\log \mathrm{Re}_U$$

for $\mathrm{Re}_U > 2\cdot 10^4$.

18.3 Energy Considerations in Pipe Flow

In Section 13.2 the Energy Grade Line (EGL) was introduced,

$$\mathrm{EGL} = \frac{p}{\rho g} + \frac{V^2}{2g} + z.$$

This can be thought of as the total mechanical energi per unit mass in a flow. This is constant for inviscid flow as there is no friction to dissipate the mechanical energy. For viscous flow, however, friction will lead to a continuously decreasing EGL.

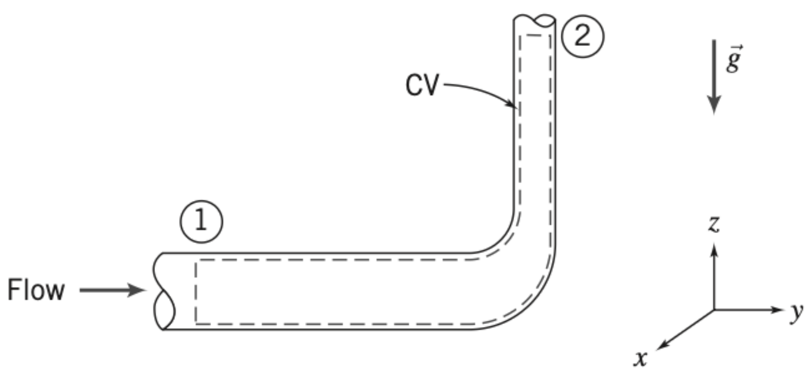


Figure 18.2: Control volume and coordinates for energy analysis through a 90° reducing elbow.

We consider the steady flow through the reducing elbow on Figure 18.2.

The first law of thermodynamics is

$$\dot{Q} - \dot{W}_s - \dot{W}_{\text{shear}} - \dot{W}_{\text{other}} = \frac{\partial}{\partial t}\int_{\text{CV}} e\rho\,\mathrm{d}V + \int_{\text{CS}} \left(e + pv\right)\rho\boldsymbol{V}\cdot\mathrm{d}\boldsymbol{A}$$

where

$$e = u + \frac{V^2}{2} + gz.$$

For the elbow we assume that no shaft work and no other work is exerted, hence $\dot{W}_s = \dot{W}_{\text{other}} = 0$. Due to the no-slip condition the velocities at the walls will be zero so no shear work will be made even though shear forces are present at the walls. Furthermore, we assume steady and incompressible flow and uniform internal energy and pressure at cross sections 1 and 2.

Under these assumptions the first law of thermodynamics for the system reduces to

$$\dot{Q} = \dot{m}\left(u_2 - u_1\right) + \dot{m}\left(\frac{p_2 - p_1}{\rho}\right) + \dot{m}g\left(z_2 - z_1\right) + \int_{A_2}\frac{V_2^2}{2}\rho V_2\,\mathrm{d}A_2 - \int_{A_1}\frac{V_1^2}{2}\rho V_1\,\mathrm{d}A_1.$$

18.3.1 Kinetic Energy Coefficient

We now define the *kinetic energy coefficient*, α, such that the product of this coefficient and the kinetic energy based on the average velocity equals the kinetic energy,

$$\int_A \frac{V^2}{2}\rho V\,\mathrm{d}A = \alpha\int_A \frac{\overline{V}^2}{2}\rho V\,\mathrm{d}A = \alpha\dot{m}\frac{\overline{V}^2}{2}$$

or

$$\alpha = \frac{\int_A \rho V^3\,\mathrm{d}A}{\dot{m}\overline{V}^2}.$$

This can in many ways be thought of as a correction factor that allows us to use the average velocity $\overline{V}$ in the energy equation to compute the kinetic energy at a given cross section.

For laminar flow in a pipe, $\alpha = 2{,}0$.

For turbulent pipe flow α can be determined as

$$\alpha = \left(\frac{U}{\overline{V}}\right)^3 \frac{2n^2}{\left(3+n\right)\left(3+2n\right)}$$

where n is the power law exponent. For "normal" $n = 7$ power profiles $\alpha = 1{,}06$. Therefore $\alpha = 1$ is oftentimes a good approximation in pipe flow calculations.

18.3.2 Head Loss

Using the definition of α the first law of thermodynamics can be written

$$\dot{Q} = \dot{m}\left(u_2 - u_1\right) + \dot{m}\left(\frac{p_2 - p_1}{\rho}\right) + \dot{m}g\left(z_2 - z_1\right) + \dot{m}\left(\frac{\alpha_2\overline{V}_2^2}{2} - \frac{\alpha_1\overline{V}_1^2}{2}\right).$$

Dividing by the mass flow rate gives

$$\frac{\delta Q}{\mathrm{d}m} = u_2 - u_1 + \frac{p_2 - p_1}{\rho} + gz_2 - gz_1 + \frac{\alpha_2\overline{V}_2^2}{2} - \frac{\alpha_1\overline{V}_1^2}{2}.$$

Rearranging this equation, we get

$$\left(\frac{p_1}{\rho}+\alpha_1\frac{\overline{V}_1^2}{2}+gz_1\right)-\left(\frac{p_2}{\rho}+\alpha_2\frac{V_2^2}{2}+gz_2\right)=(u_2-u_1)-\frac{\delta Q}{\mathrm{d}m} \tag{36}$$

In Equation (36) the term $\left(\frac{p}{\rho}+\alpha\frac{\overline{V}^2}{2}+gz\right)$ represents the mechanical energy per unit mass at a cross section. The term $u_2-u_1-\frac{\delta Q}{\mathrm{d}m}$ is the difference in mechanical energy per unit mass between sections 1 and 2. It represents the irreversible conversion of mechanical energy from section 1 to thermal energy at section 2. We identify this group of terms as the total energy loss per unit mass and designate it by h_{l_T}. Then,

$$\left(\frac{p_1}{\rho}+\alpha_1\frac{\overline{V}_1^2}{2}+gz_1\right)-\left(\frac{p_2}{\rho}+\alpha_2\frac{\overline{V}_2^2}{2}+gz_2\right)=h_{l_T} \tag{37}$$

Equation (37) allows us to compute the friction loss between two sections of pipe.

18.4 Calculation of Head Loss

Total head loss, h_{l_T}, is the sum of major losses, h_l due to frictional effects in the fully developed flow and minor losses h_{l_m} resulting from entrances, fittings, and so on.

18.4.1 Major Losses: Friction Factor

We can use the energy balance in Equation (37) to evaluate the major head loss. For fully developed flow through a constant area pipe, Equation (37) reduces to

$$\frac{p_1-p_2}{\rho}=g\left(z_2-z_1\right)+h_l \tag{38}$$

For a horizontal pipe, $z_2=z_1$, and Equation (38) reduces to

$$\frac{p_1-p_2}{\rho}=\frac{\Delta p}{\rho}=h_l \tag{39}$$

Therefore, the major head loss can be expressed as the pressure loss for fully developed flow through a horizontal pipe of constant area.

18.4.2 a. Laminar Flow

In laminar flow, it has previously been shown that the pressure drop may be computed analytically for fully developed flow in a horizontal pipe, as

$$\Delta p=32\frac{L}{D}\frac{\mu\overline{V}}{D}.$$

Substituting in Equation (39) yields

$$h_l=32\frac{L}{D}\frac{\mu\overline{V}}{\rho D}=\frac{L}{D}\frac{\overline{V}^2}{2}\left(64\frac{\mu}{\rho\overline{V}D}\right)=\frac{64}{\mathrm{Re}}\frac{L}{D}\frac{\overline{V}^2}{2} \tag{40}$$

18.4.3 b. Turbulent Flow

In turbulent flow, the pressure drop cannot be analytically evaluated and must therefore be found experimentally instead. In fully developed turbulent flow, the pressure drop Δp caused by friction in a horizontal constant-area pipe is known to depend on pipe diameter D, pipe length L, pipe roughness e, average flow velocity $\overline{V}$, fluid density ρ, and fluid viscosity μ. I.e.

$$\Delta p=\Delta p\left(D,L,e,\overline{V},\rho,\mu\right).$$

Using dimensional analysis we find a correlation of the form

$$\frac{\Delta p}{\rho \overline{V}^2} = f\left(\frac{\mu}{\rho \overline{V} D}, \frac{L}{D}, \frac{e}{D}\right).$$

We recognize $\frac{\mu}{\rho \overline{V} D} = \frac{1}{\mathrm{Re}}$ so

$$\frac{\Delta p}{\rho \overline{V}^2} = \phi\left(\mathrm{Re}, \frac{L}{D}, \frac{e}{D}\right).$$

Substituting in Equation (39) we see that

$$\frac{h_l}{V^2} = \phi\left(\mathrm{Re}, \frac{L}{D}, \frac{e}{D}\right).$$

Experimental data shows that the non-dimensional head loss is directly proportional to $\frac{L}{D}$. Hence

$$\frac{h_l}{\overline{V}^2} = \frac{L}{D}\phi_1\left(\mathrm{Re}\frac{e}{D}\right).$$

By convention $\frac{1}{2}$ is introduced in the denominator of the LHS so that the LHS is the ratio of the head loss to the kinetic energy per unit mass flow. I.e.

$$\frac{h_l}{\frac{1}{2}\overline{V}^2} = \frac{L}{D}\phi_2\left(\mathrm{Re}, \frac{e}{D}\right).$$

The unknown function $\phi_2\left(\mathrm{Re}, \frac{e}{D}\right)$ is defined as the friction factor f,

$$f \equiv \phi_2\left(\mathrm{Re}, \frac{e}{D}\right).$$

Using this,

$$h_l = f\frac{L}{D}\frac{\overline{V}^2}{2}.$$

By inserting Equation (40) into this the friction factor for laminar flow can be found to be

$$f_{\mathrm{laminar}} = \frac{64}{\mathrm{Re}}.$$

Therefore, for laminar flow, the friction factor f is solely a function of the Reynolds number.

For turbulent flow empirical correlations has led to a relation of approximately

$$\frac{1}{\sqrt{f}} = -2{,}0\log\left(\frac{\frac{e}{D}}{3{,}7} + \frac{2{,}51}{\mathrm{Re}\sqrt{f}}\right).$$

An approximation of this is

$$\frac{1}{\sqrt{f}} = -1{,}8\log\left(\left(\frac{\frac{e}{D}}{3{,}7}\right)^{1{,}11} + \frac{6{,}9}{\mathrm{Re}}\right)$$

which is accurate to within around 2 % for $\mathrm{Re} > 3000$.

For turbulent flow in smooth pipes, the Blasius correlation, valid for $\mathrm{Re} < 10^5$ is

$$f = \frac{0{,}316}{\mathrm{Re}^{0{,}25}}.$$

For $\mathrm{Re} > 10^5$ the following relation for smooth pipes is found to be accurate

$$f = \frac{0{,}184}{\mathrm{Re}^{0{,}2}}.$$

18.4.4 Minor Losses

A piping system typically has a variety of fittings, bends, etc. leading to additional head losses. These are termed *minor losses* even though they may be larger in size than the *major losses*, especially for short pipes. Minor losses are computed as

$$h_{l_m} = K\frac{\overline{V}^2}{2} \tag{41}$$

where the loss coefficient K is determined experimentally. A wide array of experimental data for different geometries are shown in Chapter 8 (~ p. 260) in the course literature.

Lecture 16: Boundary Layer Theory

October 29, 2025

20 External Incompressible Viscous Flow

20.1 The Boundary Layer Concept

The boundary layer as a concept was first introduced by Ludwig Prandtl in 1904. In many ways this served as the *common ground* between the Euler-based field of theoretical hydrodynamics, which predicted no drag and contradicted experiments in many cases, and the empirical field of *hydraulics* which was derived solely from experimental measurements.

The complete equations for viscous flow of a fluid – the Navier-Stokes equations – were already known prior to Prandtl, however these equations are impossible to solve analytically except in the simplest cases, prohibiting major theoretical results for viscous flow. Using Prandtl's idea of a boundary layer, it is possible to analyze many viscous flows by breaking these into two regions, one close to the boundary and one covering the rest of the fluid. Only in the (relatively) thin *boundary layer* closest to the boundary the effects of viscosity are important. In the bulk of the fluid (the non-boundary layer region) viscous effects are negligible and the fluid may thus often be treated as inviscid in this area.

In the boundary layer, both viscous and inertial forces are important. Consequently, the Reynolds-number is important when characterizing boundary-layer flows. The characteristic length used in the Reynolds number is either the length in the flow direction over which the boundary layer has developed or some measure of the boundary-layer thickness.

Flow in the boundary layer may either be laminar or turbulent. However, there is no unique value of the Reynolds number for which a transition from laminar to turbulent flow definitively happens in a boundary layer. Factors such as pressure gradient, surface roughness, heat transfer, body forces, and freestream disturbances also influence when this transition happen.

Often, in real flows, the boundary layer develops over a long, essentially flat surface. This is a relatively simple flow case as the flow velocity U outside the boundary layer is constant, and therefore this region is steady, inviscid, and incompressible, meaning the pressure will also be constant here. This constant pressure is the pressure felt by the boundary layer. This is also known as a *zero pressure gradient flow*.

On Figure 20.1 a qualitative picture of the growth of the boundary layer over a flat plate is shown. Here, it can be seen, that the boundary layer will be laminar for a short distance downstream from the leading edge; then, a transition will occur over a region before the flow turns fully turbulent.

For incompressible flow over a smooth plate with zero pressure gradient, in the absence of heat transfer, with negligible external disturbances this transition can be delayed until the Reynolds number, $\mathrm{Re}_x = \rho U x/\mu$ exceeds 10^6. For calculation purposes, under typical conditions, transition usually occurs at a length Reynolds number of $5 \cdot 10^5$. For air at standard conditions with free stream velocity $U = 30\,\frac{\mathrm{m}}{\mathrm{s}}$, this corresponds to $x \approx 0{,}24\,\mathrm{m}$.

As the boundary layer is defined as the area where viscous forces are present, as opposed to the free stream,

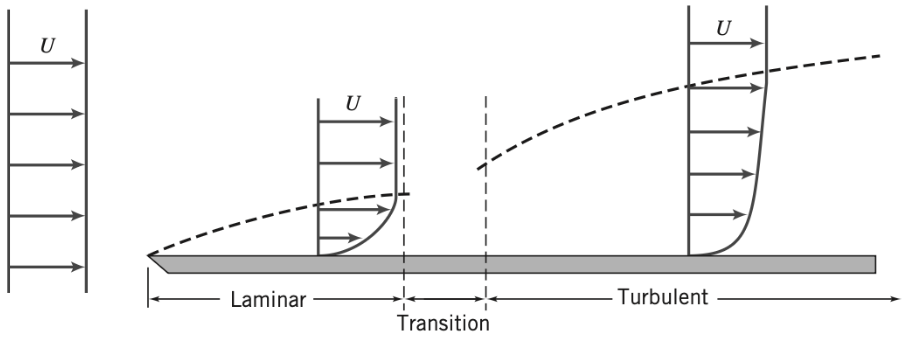
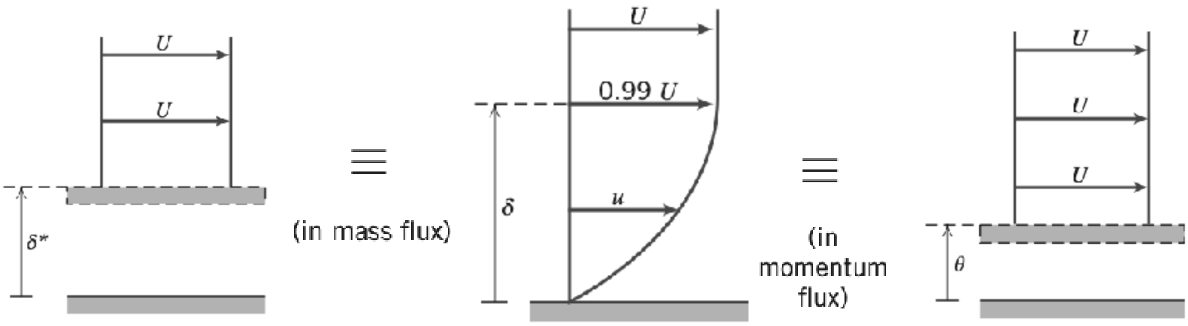


Figure 20.1: Boundary layer on a flat plate.

where viscous forces are negligible, it is not unambiguous exactly where the border between the two regions is drawn. Also, as (in the mathematically idealized situation shown on Figure 20.1) the velocity as a function of distance from the boundary layer will asymptotically approach the free stream velocity, the boundary layer cannot simply be defined as stopping at the point where the local velocity equals the free stream velocity $u = U$. Therefore, several different standards are in use such as: the boundary layer thickness δ, the displacement thickness δ^*, and the momentum thickness θ. These are all shown in comparison on Figure 20.2.



(a) Displacement thickness, δ^* (b) Disturbance thickness, δ (c) Momentum thickness, θ

Figure 20.2: Comparison between the different standards for boundary-layer thickness.

The most straightforward of these is the boundary layer thickness, δ. This is usually defined as the distance from the surface at which the difference between the velocity and the free stream velocity is within 1%, $u = 0{,}99U$.

The other two definitions both rely on the fact that the boundary layer slows the fluid, such that the mass flux and momentum flux are both less than they would be without the boundary layer present.

The displacement thickness, δ^*, is the distance the plate would be moved such that the loss of mass flux due to the reduction in flow area is equivalent to the loss the boundary layer causes. For incompressible flow this is:

$$\delta^* = \int_0^\infty 1 - \frac{u}{U}\,\mathrm{d}y \approx \int_0^\delta 1 - \frac{u}{U}\,\mathrm{d}y.$$

The momentum thickness, θ, is the distance the plate would have to be moved such that the loss of momentum flux is equivalent to the loss the boundary layer causes. This is given as:

$$\theta = \int_0^\infty \frac{u}{U}\left(1 - \frac{u}{U}\right)\mathrm{d}y \approx \int_0^\delta \frac{u}{U}\left(1 - \frac{u}{U}\right)\mathrm{d}y.$$

The displacement and momentum thickness, δ^* and θ, are easier to evaluate experimentally than the boundary layer thickness δ.

20.2 Laminar Flat Plate Boundary Layer: Exact Solution

For two-dimensional, steady, incompressible flow with zero pressure gradient the equations of motion reduces to:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$
$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}$$

with the boundary conditions

$$u(0) = 0 \qquad v(0) = 0$$
$$u(\infty) = U \qquad \frac{\partial u}{\partial y}(\infty) = 0.$$

These constitute a set of nonlinear, coupled, PDE's for the unknown velocity field u and v. In 1908, H. Blasius showed that the solution to these is of the form

$$\frac{u}{U} = g(\eta) \quad \text{where} \quad \eta \propto \frac{y}{\delta}.$$

Further, Blasius reasoned that $\delta \propto \sqrt{\nu x / U}$ and got

$$\eta = y\sqrt{\frac{U}{\nu x}} \tag{42}$$

The variable η combines the variables x and y into one variable. Using the stream function one can define the function

$$f(\eta) = \frac{\psi}{\sqrt{\nu x U}}.$$

The exact solution will not be given in these notes as it is not expected that the derivation will be relevant on an exam. The solution of this in terms of η is shown on Figure 20.3. From Figure 20.3 we see that at $\eta = 5{,}0$, $u/U \approx 0{,}99$. Therefore from Equation (42) we get:

$$\delta \approx \frac{5{,}0}{\sqrt{\frac{U}{\nu x}}} = \frac{5{,}0x}{\sqrt{\mathrm{Re}_x}}.$$

The wall shear stress is:

$$\tau_w = 0{,}332U\sqrt{\rho\mu\frac{U}{x}} = \frac{0{,}332\rho U^2}{\sqrt{\mathrm{Re}_x}}$$

and the wall shear stress coefficient, C_f, is given by

$$C_f = \frac{\tau_w}{\frac{1}{2}\rho U^2} = \frac{0{,}664}{\sqrt{\mathrm{Re}_x}}. \tag{43}$$

20.3 Momentum Integral Equation

We consider the situation of an incompressible, steady, two-dimensional flow over a solid surface. The boundary-layer thickness, δ, grows with increasing distance, x. We choose a differential control volume of length $\mathrm{d}x$, width w, and height $\delta(x)$, as shown in Figure 20.4. The free stream velocity is denoted $U(x)$.

We will seek to determine the boundary-layer thickness and shear stress as functions of x. There will be mass flow across surfaces ab and cd of volume $abcd$. As surface bc is not a streamline but the *imaginary* boundary

$\eta = y\sqrt{\frac{U}{\nu x}}$	f	$f' = \frac{u}{U}$	f''
0	0	0	0.3321
0.5	0.0415	0.1659	0.3309
1.0	0.1656	0.3298	0.3230
1.5	0.3701	0.4868	0.3026
2.0	0.6500	0.6298	0.2668
2.5	0.9963	0.7513	0.2174
3.0	1.3968	0.8460	0.1614
3.5	1.8377	0.9130	0.1078
4.0	2.3057	0.9555	0.0642
4.5	2.7901	0.9795	0.0340
5.0	3.2833	0.9915	0.0159
5.5	3.7806	0.9969	0.0066
6.0	4.2796	0.9990	0.0024
6.5	4.7793	0.9997	0.0008
7.0	5.2792	0.9999	0.0002
7.5	5.7792	1.0000	0.0001
8.0	6.2792	1.0000	0.0000

Figure 20.3: The function $f(\eta)$ for the Laminar Boundary Layer along a flat plate at Zero incidence.

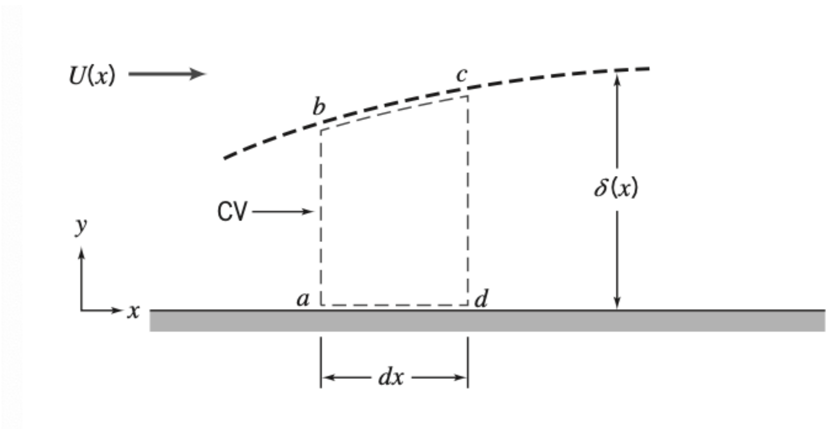


Figure 20.4: Differential control volume in a boundary layer.

separating the boundary layer from the free stream there will also be mass flow across bc. As ad constitutes a solid boundary no mass flow will exist across this boundary. Using the continuity equation for steady flow we start by determining the mass flux through each portion of the control surface.

$$\int_{\text{CS}} \rho \boldsymbol{V} \cdot \mathrm{d}\boldsymbol{A} = 0.$$

Hence

$$\dot{m}_{ab} + \dot{m}_{bc} + \dot{m}_{cd} = 0 \quad \Longrightarrow \quad \dot{m}_{bc} = -(\dot{m}_{ab} + \dot{m}_{cd}).$$

The mass flux through ab is simply

$$\dot{m}_{ab} = -\left(\int_0^{\delta} \rho u \,\mathrm{d}y\right) w.$$

Using a Taylor-series expansion the mass flux through cd can be determined to be

$$\dot{m}_{cd} = \left(\int_0^{\delta} \rho u \,\mathrm{d}y + \frac{\partial}{\partial x}\left(\int_0^{\delta} \rho u \,\mathrm{d}y\right) \mathrm{d}x\right).$$

Thus for surface $\dot{m}_{bc}$ using the above results and the continuity equation we get:

$$\dot{m}_{bc} = -\left(\frac{\partial}{\partial x}\left(\int_0^\delta \rho u \, \mathrm{d}y\right)\mathrm{d}x\right)w.$$

Now we will apply the Momentum equation to the control volume $abcd$, remembering that the flow is steady and under the assumption that body forces are negligible.

We get:

$$(F_S)_x = \mathrm{mf}_{ab} + \mathrm{mf}_{bc} + \mathrm{mf}_{cd}$$

where mf represents the x component of momentum flux.

Once again, we consider the momentum flux over each surface separately. For surface ab we simply get

$$\mathrm{mf}_{ab} = -\left(\int_0^\delta u\rho u \, \mathrm{d}y\right)w.$$

Using a Taylor series about x we get the momentum flux through cd mf_{cd} as

$$\mathrm{mf}_{cd} = \left(\int_0^\delta u\rho u \, \mathrm{d}y + \frac{\partial}{\partial x}\left(\int_0^\delta u\rho u \, \mathrm{d}y\right)\mathrm{d}x\right)w.$$

Since the mass crossing surface bc has velocity component U in the x direction, the x momentum flux across bc is

$$\begin{aligned}
\mathrm{mf}_{bc} &= U\dot{m}_{bc} \\
\mathrm{mf}_{bc} &= -U\left(\frac{\partial}{\partial x}\left(\int_0^\delta \rho u \, \mathrm{d}y\right)\mathrm{d}x\right)w.
\end{aligned}$$

Using these we can calculate the net x momentum flux through the control surface as:

$$\int_{\mathrm{CS}} u\rho \boldsymbol{V} \cdot \mathrm{d}\boldsymbol{A} = \left(\frac{\partial}{\partial x}\left(\int_0^\delta u\rho u \, \mathrm{d}y\right)\mathrm{d}x - U\frac{\partial}{\partial x}\left(\int_0^\delta \rho u \, \mathrm{d}y\right)\mathrm{d}x\right)w.$$

Now we have an expression for the momentum flux in the x direction through the control surface and we therefore simply need to find expressions for the surface forces acting on the control volume. Note that ab, bc, and cd all experience normal forces (i.e. pressure) that generates a force in the x direction. In addition, a shear force acts on ad. Since the velocity gradient goes to zero at the edge of the boundary layer, the shear force acting along surface bc is negligible.

If the pressure at x is p, then the force acting on ab is

$$F_{ab} = pw\delta.$$

Due to the small size of the boundary layer it is assumed that $p = p(x)$ within the boundary layer.

Using a Taylor series the force on cd is found to be

$$F_{cd} = -\left(p + \left.\frac{\partial p}{\partial x}\right]_x \mathrm{d}x\right)w\left(\delta + \delta\right).$$

The force on bc is

$$F_{bc} = \left(p + \frac{1}{2}\left.\frac{\partial p}{\partial x}\right]_x \mathrm{d}x\right)w\,\mathrm{d}\delta.$$

The shear force acting on ad is

$$F_{ad} = -\left(\tau_w + \frac{1}{2}\,\mathrm{d}\tau_w\right) w\,\mathrm{d}x.$$

Summing these we obtain the total force acting in the x direction on the control volume

$$(F_S)_x \left(-\frac{\partial p}{\partial x}\delta\,\mathrm{d}x - \frac{1}{2}\frac{\partial p}{\partial x}\,\mathrm{d}x\,\mathrm{d}\delta - \tau_w\,\mathrm{d}x - \frac{1}{2}\,\mathrm{d}\tau_w\,\mathrm{d}x\right) w.$$

As $\mathrm{d}x\,\mathrm{d}\delta \ll \delta\,\mathrm{d}x$ and $\mathrm{d}\tau_w \ll \tau_w$ we can neglect the second and fourth terms, leaving us with:

$$(F_S)_x = \left(-\frac{\partial p}{\partial x}\delta\,\mathrm{d}x - \tau_w\,\mathrm{d}x\right) w.$$

Now substituting these expressions for $\int_{\mathrm{CS}} u\rho \boldsymbol{V}\cdot \mathrm{d}\boldsymbol{A}$ and $(F_S)_x$ into the momentum equation we obtain

$$\left(-\frac{\partial p}{\partial x}\delta\,\mathrm{d}x - \tau_w\,\mathrm{d}x\right) w = \left(\frac{\partial}{\partial x}\left(\int_0^\delta u\rho u\,\mathrm{d}y\right)\mathrm{d}x - U\frac{\partial}{\partial x}\left(\int_0^\delta \rho u\,\mathrm{d}y\right)\mathrm{d}x\right) w.$$

Dividing this by $w\,\mathrm{d}x$ gives

$$-\delta\frac{\partial p}{\partial x} - \tau_w = \frac{\partial}{\partial x}\int_0^\delta u\rho u\,\mathrm{d}y - U\frac{\partial}{\partial x}\int_0^\delta \rho u\,\mathrm{d}y.$$

The above can be rewritten as

$$\tau_w = -\frac{\partial}{\partial x}\int_0^\delta u\rho u\,\mathrm{d}y + U\frac{\partial}{\partial x}\int_0^\delta \rho u\,\mathrm{d}y + \frac{\mathrm{d}U}{\mathrm{d}x}\int_0^\delta \rho U\,\mathrm{d}y.$$

Or, rewriting further and using the definitions of displacement thickness δ^* and momentum thickness θ, we obtain:

$$\frac{\tau_w}{\rho} = \frac{\mathrm{d}}{\mathrm{d}x}\left(U^2\theta\right) + \delta^* U\frac{\mathrm{d}U}{\mathrm{d}x} \tag{44}$$

Equation (44) is known as the *momentum integral equation* and is restricted to a steady incompressible two-dimensional flow. This will yield an ODE for boundary-layer thickness δ as a function of x.

20.4 Use of the Momentum Integral Equation for Flow with Zero Pressure Gradient

For the special case of a flat plate (i.e. zero pressure gradient) the free stream pressure p and velocity U are both constant.

The momentum integral equation then reduces to

$$\tau_w = \rho U^2\frac{\mathrm{d}\theta}{\mathrm{d}x} = \rho U^2\frac{\mathrm{d}}{\mathrm{d}x}\int_0^\delta \frac{u}{U}\left(1-\frac{u}{U}\right)\mathrm{d}y.$$

Here, the velocity distribution u/U in the boundary layer is assumed to be similar for all x and is normally given as a function of y/δ. For this reason it is convenient to change the variable of integration from y to y/δ. By defining

$$\eta = \frac{y}{\delta}$$

we get

$$\mathrm{d}y = \delta\,\mathrm{d}\eta$$

and the momentum integral equation for zero pressure gradient is written

$$\tau_w = \rho U^2\frac{\mathrm{d}\theta}{\mathrm{d}x} = \rho U^2\frac{\mathrm{d}\delta}{\mathrm{d}x}\int_0^1 \frac{u}{U}\left(1-\frac{u}{U}\right)\mathrm{d}\eta.$$

To solve this equation for the boundary layer thickness as a function of x we first assume a velocity distribution in the boundary layer of the form

$$\frac{u}{U} = f\left(\frac{y}{\delta}\right).$$

The assumed velocity distribution should satisfy the boundary conditions:

$$\begin{aligned} u(0) &= 0 \\ u(\delta) &= U \\ \frac{\mathrm{d}u}{\mathrm{d}y}(\delta) &= 0. \end{aligned}$$

Once a velocity distribution is assumed the integral in the equation from before reduces to:

$$\int_0^1 \frac{u}{U}\left(1 - \frac{u}{U}\right) \mathrm{d}\eta = \frac{\eta}{\delta} = \text{constant} = \beta$$

and the momentum integral equation becomes

$$\tau_w = \rho U^2 \frac{\mathrm{d}\delta}{\mathrm{d}x}\beta,$$

which is an expression for τ_w in terms of δ. This can be used to solve for $\delta(x)$.

Laminar Flow For laminar flow over a flat plate, it turns out that a reasonable assumption for the velocity profile is a polynomial in y:

$$u = a + by + cy^2.$$

Evaluating constants a, b, and c given the boundary conditions we obtain:

$$\frac{u}{U} = 2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2 = 2\eta - \eta^2.$$

The wall shear stress is given by:

$$\tau_w = \mu \frac{\partial u}{\partial y}\bigg)_{y=0}.$$

Substituting the assumed velocity profile into this we get:

$$\tau_w = \mu \frac{\partial u}{\partial y}\bigg]_{y=0} = \mu \frac{U \partial \frac{u}{U}}{\delta \partial \frac{y}{\delta}}_{\frac{y}{\delta}=0} = \frac{\mu U}{\delta} \frac{\mathrm{d}\frac{u}{U}}{\mathrm{d}\eta}\bigg]_{\eta=0}$$

or

$$\tau_w = \frac{\mu U}{\delta} \frac{\mathrm{d}}{\mathrm{d}\eta}\left(2\eta - \eta^2\right)\bigg]_{\eta=0} = \frac{\mu U}{\delta}\left(2 - 2\eta\right)\bigg]_{\eta=0} = \frac{2\mu U}{\delta}.$$

This shows that the wall stress τ_w is a function of x, since the boundary layer thickness $\delta = \delta(x)$.

Substituting for τ_w and u/U into the momentum integral equation Equation (44), we obtain

$$\frac{2\mu U}{\delta} = \rho U^2 \frac{\mathrm{d}\delta}{\mathrm{d}x} \int_0^1 \left(2\eta - \eta^2\right)\left(1 - 2\eta + \eta^2\right) \mathrm{d}\eta$$

or

$$\frac{2\mu U}{\delta \rho U^2} = \frac{\mathrm{d}\delta}{\mathrm{d}x} \int_0^1 \left(2\eta - 5\eta^2 + 4\eta^3 - \eta^4\right) \mathrm{d}\eta.$$

Integrating this and substituting limits yields

$$\frac{2\mu}{\delta \rho U} = \frac{2}{15} \frac{\mathrm{d}\delta}{\mathrm{d}x} \quad \text{or} \quad \delta \, \mathrm{d}\delta = \frac{15\mu}{\rho U} \mathrm{d}x$$

which is a differential equation for δ. Integrating again yields

$$\frac{\delta^2}{2} = \frac{15\mu}{\rho U} x + c.$$

We assume $\delta = 0$ at $x = 0$, meaning $c = 0$ and thus

$$\delta = \sqrt{\frac{30\mu x}{\rho U}}.$$

This shows that a laminar boundary layer will grow in thickness δ with $\sqrt{x}$; i.e. it has a parabolic shape. Traditionally this is expressed in dimensionless form:

$$\frac{\delta}{x} = \sqrt{\frac{30\mu}{\rho U x}} = \frac{5{,}48}{\sqrt{\mathrm{Re}_x}}.$$

I.e. the ratio of boundary layer thickness to distance along a flat plate varies inversely with the square root of length Reynolds number.

The wall shear stress, or "skin friction" coefficient is defined as:

$$C_f \equiv \frac{\tau_w}{\frac{1}{2}\rho U^2}.$$

Substituting in the velocity profile and the equation for δ / x we get:

$$C_f = \frac{\tau_w}{\frac{1}{2}\rho U^2} = \frac{2\mu\frac{U}{\delta}}{\frac{1}{2}\rho U^2} = \frac{4\mu}{\rho U \delta} = \frac{0{,}730}{\sqrt{\mathrm{Re}_x}}.$$

Turbulent Flow For the flat plate, we still have U = constant. We must, once again, start by approximating the velocity profile and an expression for τ_w. The turbulent velocity profile is often assumed to be

$$\frac{u}{U} = \left(\frac{y}{\delta}\right)^{\frac{1}{7}} = \eta^{\frac{1}{7}}.$$

However, this does not hold right next to the wall as it predicts $\mathrm{d}u/\mathrm{d}y(0) = \infty$. Consequently, it cannot be used to define τ_w. We therefore adapt the expression for pipe flow to:

$$\tau_w = 0{,}0332\rho \overline{V}^2 \left(\frac{\nu}{R\overline{V}}\right)^{0{,}25}.$$

For a 1/7-power profile in a pipe, $\overline{V}/U = 0{,}817$. Substituting this and $R = \delta$ into the above we get

$$\tau_w = 0{,}0233\rho U^2 \left(\frac{\nu}{U\delta}\right)^{\frac{1}{4}}.$$

Substituting this into Equation (44) we get:

$$0{,}0233\left(\frac{\nu}{U\delta}\right)^{\frac{1}{4}} = \frac{\mathrm{d}\delta}{\mathrm{d}x}\int_0^1 \eta^{\frac{1}{7}}\left(1-\eta^{\frac{1}{7}}\right)\mathrm{d}\eta = \frac{7}{72}\frac{\mathrm{d}\delta}{\mathrm{d}x}.$$

This gives a differential equation for δ:

$$\delta^{\frac{1}{4}}\,\mathrm{d}\delta = 0{,}240\left(\frac{\nu}{U}\right)^{\frac{1}{4}}\mathrm{d}x.$$

By integration we get:

$$\frac{4}{5}\delta^{\frac{5}{4}} = 0{,}240\left(\frac{\nu}{U}\right)^{\frac{1}{4}} x + c.$$

We assume that $\delta \approx 0$ at $x = 0$, meaning the flow is assumed turbulent from the leading edge, then $c = 0$ and

$$\delta = 0{,}382\left(\frac{\nu}{U}\right)^{\frac{1}{5}} x^{\frac{4}{5}}.$$

This shows that the turbulent boundary layer thickness δ grows as $x^{\frac{4}{5}}$. In dimensionless form this is:

$$\frac{\delta}{x} = 0{,}382\left(\frac{\nu}{Ux}\right)^{\frac{1}{5}} = \frac{0{,}382}{\mathrm{Re}_x^{\frac{1}{5}}}.$$

And the skin friction coefficient is:

$$C_f = \frac{\tau_w}{\frac{1}{2}\rho U^2} = \frac{0{,}0594}{\mathrm{Re}_x^{\frac{1}{5}}} \tag{45}$$

Experiments show that Equation (45) is fairly accurate for $5 \cdot 10^5 < \mathrm{Re}_x < 10^7$.

20.5 Pressure Gradients in Boundary Layer Flow

For all bodies, except a flat plate as previously studied, a pressure gradient will be present.

A *favorable pressure gradient* is one in which the pressure decreases in the flow direction ($\partial p/\partial x < 0$). This tends to work to overcome friction-caused slowing in the boundary layer. On the other hand, an *adverse pressure gradient* is one in which pressure increases in the flow direction ($\partial p/\partial x > 0$). If an adverse pressure gradient is severe enough, the fluid particles in the boundary layer will be brought to rest. When this occurs, the particles will be forces away from the body surface (a phenomenon called *flow separation*), which leads to a turbulent wake.

For uniform flow over a flat plate, the flow *never* separates and we never develop a wake region, whether the boundary layer is laminar or turbulent, regardless of plate length.

For favorable pressure gradients we will also never observe flow separation. Furthermore, an adverse pressure gradient does not *always* lead to flow separation and a wake – i.e., it is a necessary but not a sufficient condition for flow separation.

Lecture 17: Fluid Flows About Immersed Bodies – Drag, Lift and Airfoils — November 3, 2025

Whenever a body moves relatively to a viscous fluid surrounding it, the body will experience a net force $\boldsymbol{F}$. The magnitude of this force depends on factors such as relative velocity $\boldsymbol{V}$, body shape, body size, fluid properties and so forth. As the fluid flows around the immersed body, it will generate surface stresses on each element of the surface, resulting in the aforementioned force. The surface stresses are composed of a tangential component due to viscous action and a normal component due to pressure action.

21.1 Drag

Drag is the component of force on a body acting parallel to the direction of relative motion (i.e. left/right for a 1 dimensional body moving left to right). In Equation (25) a functional form of the drag force F_D was given as

$$F_D = f_1(d, V, \mu, \rho).$$

Application of the Buckingham Pi theorem resulted in two dimensionless Π parameters that were written in functional form as

$$\frac{F_D}{\rho V^2 d^2} = f_2\left(\frac{\rho V d}{\mu}\right) = f_2(\mathrm{Re}).$$

As d^2 is proportional to cross sectional area we could also write

$$\frac{F_D}{\rho V^2 A} = f_3\left(\frac{\rho V d}{\mu}\right) = f_3\,(\mathrm{Re}) \tag{46}$$

This is valid for incompressible flow over *any* body.

The *drag coefficient*, C_D, is defined as

$$C_D \equiv \frac{F_D}{\frac{1}{2}\rho V^2 A}.$$

Using this Equation (46) can be written as

$$C_D = f\,(\mathrm{Re}).$$

If compressibility and free-surface effects are also included in this the following functional form is obtained instead

$$C_D = f\,(\mathrm{Re}, \mathrm{Fr}, M).$$

21.1.1 Pure Friction Drag: Flow over a Flat Plate Parallel to the Flow

This is the same flow situation as was discussed in Section 20.4. Since the pressure gradient is zero and pressure forces that are perpendicular to the plate do not contribute to the drag, the total drag is equal to the friction drag. This

$$F_D = \int_{\text{plate surface}} \tau_w \,\mathrm{d}A$$

and

$$C_D = \frac{F_D}{\frac{1}{2}\rho V^2 A} = \frac{\int_{\mathrm{PS}} \tau_w \,\mathrm{d}A}{\frac{1}{2}\rho V^2 A} \tag{47}$$

where A is the total surface area in contact with the fluid. The drag coefficient for a flat plate parallel to the flow depends on the shear stress distribution along the plate.

For laminar flow over a flat plate, the shear stress coefficient was, in Equation (43) given as

$$C_f = \frac{\tau_w}{\frac{1}{2}\rho U^2} = \frac{0{,}664}{\sqrt{\mathrm{Re}_x}}.$$

The drag coefficient C_D for flow with free stream velocity V, over a flat plate of length L and width b is then obtained by substituting τ_w from Equation (43) into Equation (47) as

$$\begin{aligned}
C_D &= \frac{1}{A}\int_A 0{,}664\mathrm{Re}_x^{-0{,}5}\,\mathrm{d}A \\
&= \frac{1}{bL}\int_0^L 0{,}664\left(\frac{V}{\nu}\right)^{-0{,}5} x^{-0{,}5}\, b\,\mathrm{d}x \\
&= \frac{0{,}664}{L}\left(\frac{\nu}{V}\right)^{0{,}5}\left[\frac{x^{0{,}5}}{0{,}5}\right]_0^L \\
&= 1{,}33\left(\frac{\nu}{VL}\right)^{0{,}5} \\
&= \frac{1{,}33}{\sqrt{\mathrm{Re}_L}}.
\end{aligned}$$

Assuming the boundary layer is turbulent from the leading edge, the shear stress coefficient, as given by the numerical results from Figure 20.4, is given by

$$C_f = \frac{\tau_w}{\frac{1}{2}\rho U^2} = \frac{0{,}0594}{\mathrm{Re}_x^{\frac{1}{5}}}. \tag{48}$$

Substituting τ_w from Equation (48) into Equation (47) we get

$$\begin{aligned}
C_D &= \frac{1}{A}\int_A 0{,}0594\mathrm{Re}_x^{-0{,}2}\,\mathrm{d}A \\
&= \frac{1}{bL}\int_0^L 0{,}0594\left(\frac{V}{\nu}\right)^{-0{,}2} x^{-0{,}2}\,b\,\mathrm{d}x \\
&= \frac{0{,}0594}{L}\left(\frac{\nu}{V}\right)^{0{,}2}\left[\frac{x^{0{,}8}}{0{,}8}\right]_0^L \\
&= 0{,}0742\left(\frac{\nu}{VL}\right)^{0{,}2} \\
&= \frac{0{,}0742}{\mathrm{Re}_L^{\frac{1}{5}}}.
\end{aligned}$$

The above result is valid for $5\cdot10^5 < \mathrm{Re}_L < 10^7$.

For $Re_L < 10^9$ an empirical result has been found to be

$$C_D = \frac{0{,}455}{(\log \mathrm{Re}_L)^{2{,}58}}.$$

For a boundary layer that is initially laminar and undergoes transition at some location on the plate, the turbulent drag coefficient must be adjusted to account for laminar flow over the initial length. This adjustment is made by subtracting B/Re_L from the C_D determined for completely turbulent flow. The value of B depends on the Reynolds number at transition, as

$$B = \mathrm{Re}_{\mathrm{tr}}\left(C_{D_{\mathrm{turbulent}}} - C_{D_{\mathrm{laminar}}}\right).$$

For a transition Reynolds number of $5\cdot10^5$, the drag coefficient may be calculated by adjusting either of the formulae for the turbulent drag coefficients as:

$$\begin{aligned}
C_D &= \frac{0{,}0742}{\mathrm{Re}_L^{\frac{1}{5}}} - \frac{1740}{\mathrm{Re}_L} && \left(5\cdot10^5 < \mathrm{Re}_L < 10^7\right) \\
C_D &= \frac{0{,}455}{(\log \mathrm{Re}_L)^{2{,}58}} - \frac{1610}{\mathrm{Re}_L} && \left(5\cdot10^5 < \mathrm{Re}_L < 10^7\right).
\end{aligned}$$

The variation of the drag coefficient for a flat plate parallel to the flow is shown on Figure 21.1. On Figure 21.1, transition was assumed to occur at $\mathrm{Re}_x = 5\cdot10^5$ for flows in which the boundary layer was initially laminar. The actual Reynolds number for which this transition happens depends on a variety of factors such as surface roughness, disturbances, etc.

21.1.2 Pure Pressure Drag: Flow over a Flat Plate Normal to the Flow

In flow over a flat plate normal to the flow, as shown on Figure 21.2, the wall shear stress is perpendicular to the flow direction and therefore does not contribute to the drag force. The drag is given by

$$F_D = \int_{\mathrm{surface}} \rho\,\mathrm{d}A.$$

For this geometry, the flow separates from the edges of the plate; i.e. there is a back-flow in the low energy wake of the plate. This means that although the pressure at the rear surface of the plate is essentially constant, its magnitude cannot be determined analytically. Therefore, we must resort to experiments to determine the drag force.

The drag coefficient for flow over an immersed object is usually based on the frontal area of the object. The drag

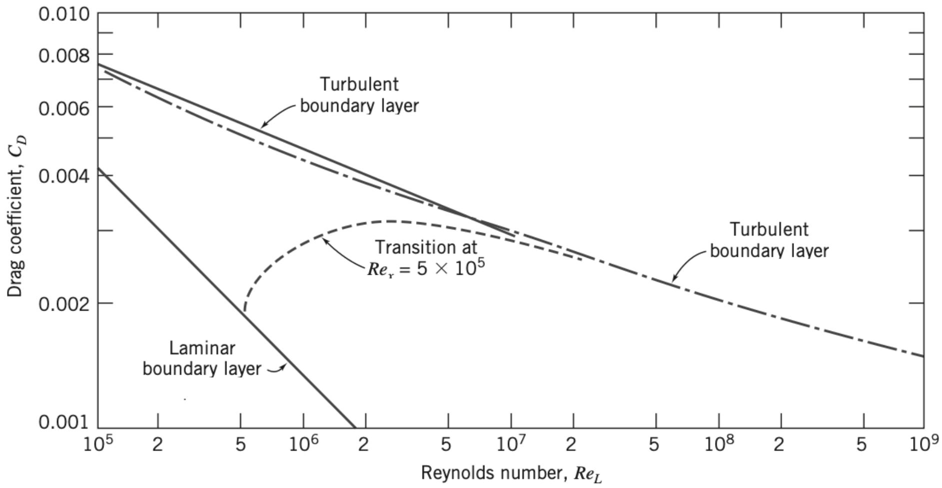


Figure 21.1: Variation of drag coefficient with Reynolds number for a smooth flat plate parallel to the flow.

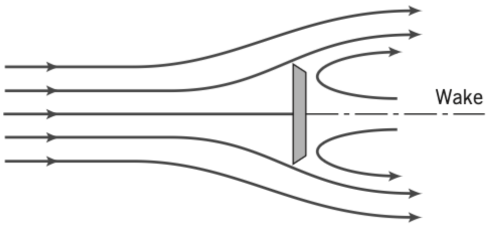


Figure 21.2: Flow over a flat plate normal to the flow

coefficient for a finite plate normal to the flow depends on the ratio of plate width to height and on the Reynolds number. For Re (based on height) greater than about 1000, the drag coefficient is essentially independent of the Reynolds number. The variation of C_D with the ratio of plate width to height b/h is shown on Figure 21.3. For $b/h = 1{,}0$ the drag coefficient is a minimum at $C_D = 1{,}18$, which is just slightly higher than for a circular disk with $C_D = 1{,}17$ at large Reynolds number.

For objects with sharp edges, the drag coefficient is essentially independent of Reynolds number (for $Re \geq 1000$). On Figure 21.4 drag coefficients for selected objects for $\mathrm{Re} \geq 10^3$ is shown.

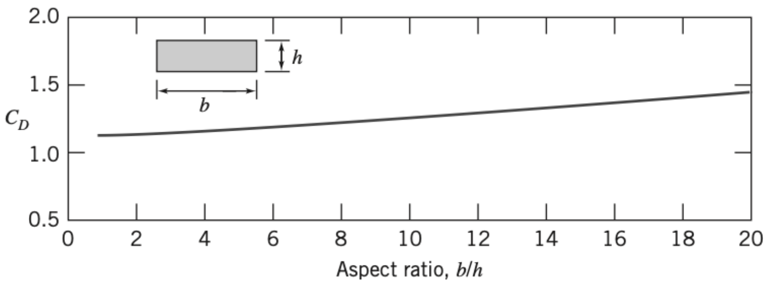
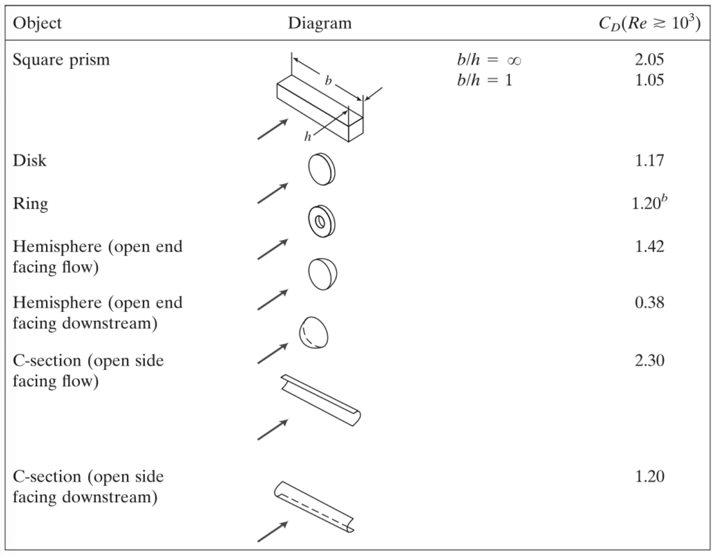


Figure 21.3: Variation of drag coefficient with aspect ratio for a flat finite plate normal to the flow with $\mathrm{Re}_h > 1000$.

21.1.3 Friction and Pressure Drag: Flow over a Sphere and Cylinder

We have looked at two special flow cases, in which either friction or pressure drag was the sole form of drag present. For friction-dominated drag, the drag coefficient was highly dependent on Reynolds number and for pressure-dominated drag the drag coefficient was largely independent of Reynolds number for $\mathrm{Re} \geq 1000$.

For flow over a sphere, both friction and pressure drag contribute noteworthily to the total drag. The drag



Object	Diagram	$C_D(Re \gtrsim 10^3)$
Square prism	$b/h = \infty$	2.05
	$b/h = 1$	1.05
Disk		1.17
Ring		1.20^b
Hemisphere (open end facing flow)		1.42
Hemisphere (open end facing downstream)		0.38
C-section (open side facing flow)		2.30
C-section (open side facing downstream)		1.20

Figure 21.4: Drag Coefficient Data for Selected Objects for Re ≥ 10^3

coefficient for flow over a smooth sphere as a function of Reynolds number is depicted on Figure 21.5.

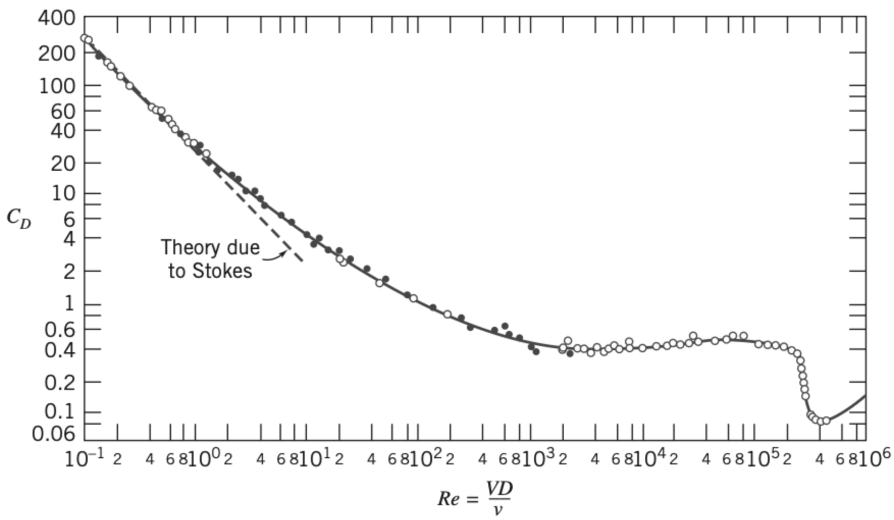


Figure 21.5: Drag coefficient of a smooth sphere as a function of Reynolds number.

At very low Reynolds number, Re ≤ 1, there is no flow separation from a sphere; i.e. the wake is laminar and friction-drag will therefore dominate. For very low Reynolds number flows, where inertia forces may be neglected, the drag force on a sphere of diameter d, moving at speed V through a fluid of viscosity μ is

$$F_D = 3\mu\pi V d.$$

The drag coefficient, C_D is then

$$C_D = \frac{24}{\text{Re}}.$$

As shown on Figure 21.5 this expression agrees with experimental values at low Reynolds numbers, but it begins to deviate significantly from the data for Re > 1,0.

As the Reynolds number increases further, the drag coefficient drops up to a Reynolds number of about 1000. A turbulent wake develops and grows as the separation point moves from the rear of the sphere toward the front.

This wake leads to a relatively large amount of pressure drag. At Re ≈ 1000, about 95% of the total drag is due to pressure. For $10^3 < \mathrm{Re} < 3 \cdot 10^5$, the drag coefficient is approximately constant. In this range the entire sphere has a low-pressure turbulent wake, as indicated on Figure 21.6. Note that $C_D \propto 1/\mathrm{Re}$ corresponds to $F_D \propto V$, and that $C_D \sim$ constant corresponds to $F_D \propto V^2$, indicating a rapid increase in drag.

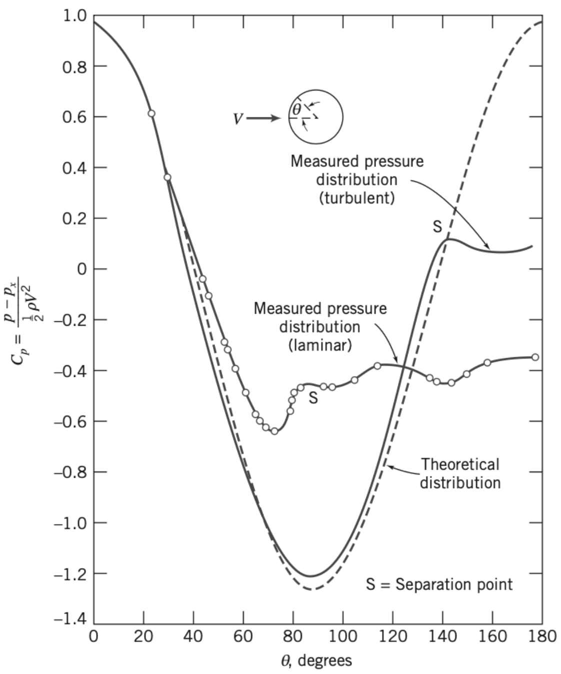


Figure 21.6: Pressure distribution around a smooth sphere for laminar and turbulent boundary-layer flow, compared with inviscid flow.

For Reynolds numbers larger than about $3 \cdot 10^5$, transition occurs and the boundary on the forward portion of the sphere becomes turbulent, which in turn leads to a decrease in the size of the wake, leading to an abrupt decrease in drag coefficient.

A turbulent boundary layer can, due to its higher momentum flux, better resist an adverse pressure gradient. Consequently, a turbulent boundary-layer is desirable on a blunt body as it delays separation and in turn reduces pressure drag.

The drag coefficient for a sphere with a turbulent boundary layer is about 1/5 that for laminar flow near the critical Reynolds number. This is the reason golf balls are equipped with dimples.

21.2 Lift

For most objects in motion through a fluid the most significant force is drag. However for some objects, such as airfoils, lift effects are also significant. Lift is defined as the component of fluid force perpendicular to the fluid motion. For an airfoil, the lift coefficient, C_L, is defined as

$$C_L \equiv \frac{F_L}{\frac{1}{2}\rho V^2 A_p}.$$

The lift and drag coefficients for an airfoil are functions of both Reynolds number and angle of attack (AoA). The AoA, α, is the angle between the airfoil chord and the free stream velocity vector. The *chord* of an airfoil is the straight line joining its leading and trailing edges.

Airfoil stall results when flow separation occurs over a major portion of the upper surface of the airfoil. As the AoA is increased, the stagnation point moves back along the lower surface of the airfoil as shown schematically for the symmetric laminar-flow section on Figure 21.7. Flow on the upper surface must then accelerate sharply to round the nose of the airfoil. The effect of AoA on the theoretical upper surface pressure distribution is shown on Figure 21.8. The minimum pressure becomes lower, and its location moves forward on the upper surface. A severe pressure gradient will appear following the point of minimum pressure, which causes the flow to separate completely from the upper surface and the airfoil stalls.

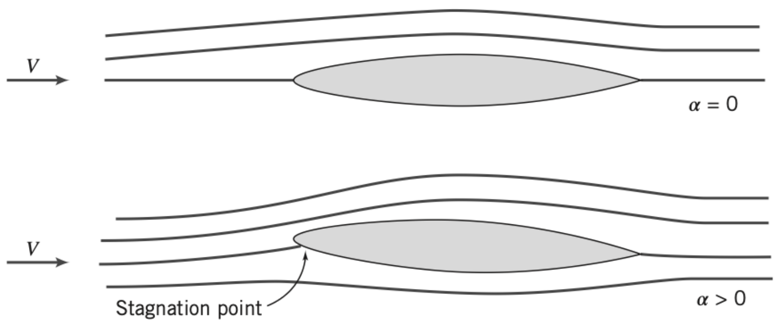


Figure 21.7: Effects of angle of attack on flow pattern.

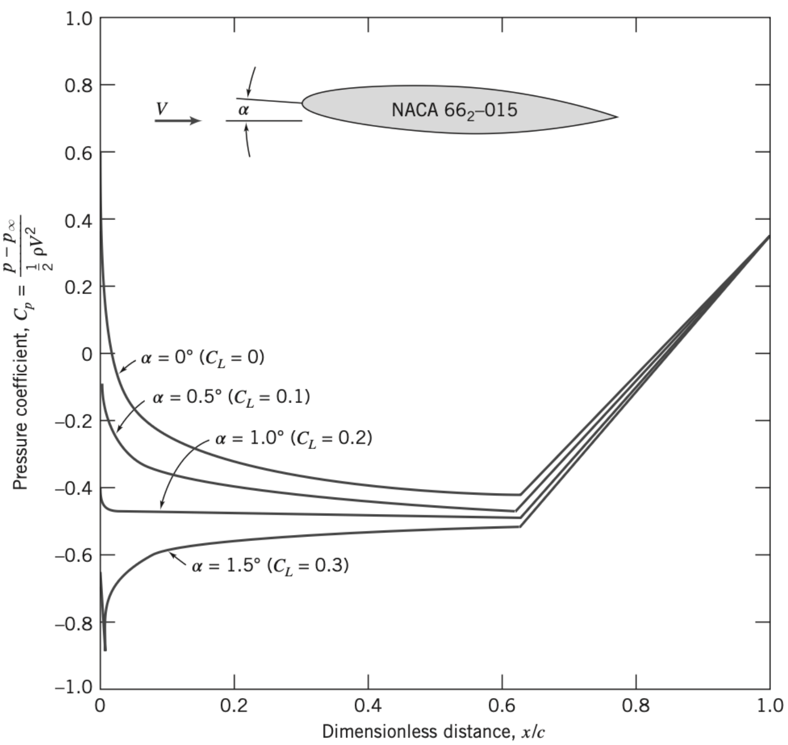


Figure 21.8: Effect of angle of attack on theoretical pressure distribution for a symmetric laminar flow airfoil of 15% thickness ratio.

Stall is indicated on Figure 21.9 as the AoA for which the lift coefficient decreases sharply.

Movement of the minimum pressure point and increase in the adverse pressure gradient are responsible for the sudden increase in C_D for the laminar flow section on Figure 21.10. The sudden rise in C_D is caused by transition from laminar to turbulent boundary layer flow on the upper surface. Aircrafts are designed to cruise in the low drag region.

Because laminar flow sections have very sharp leading edges, all of the described effects are exaggerated, and they will stall at lower AoA's than conventional sections, as shown in Figures 21.9 and 21.10.

Plots of C_L versus C_D (called *lift-drag polars*) are often used to present airfoil data in compact form. In general a

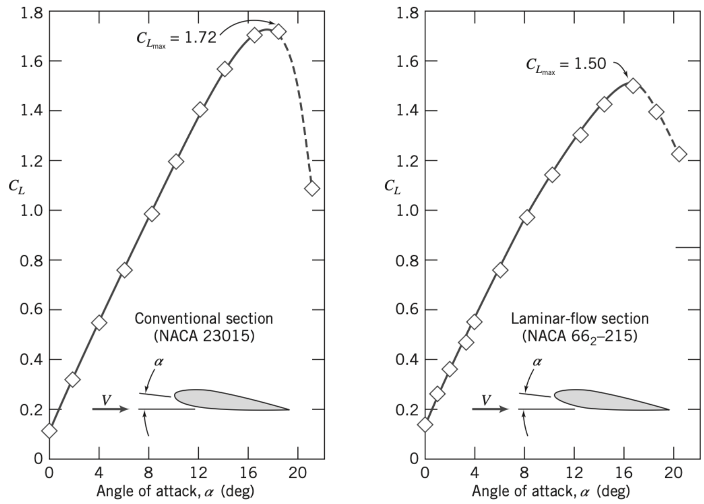


Figure 21.9: Lift coefficient versus angle of attack for two airfoil sections of 15% thickness ratio at $\text{Re}_c = 9 \cdot 10^6$.

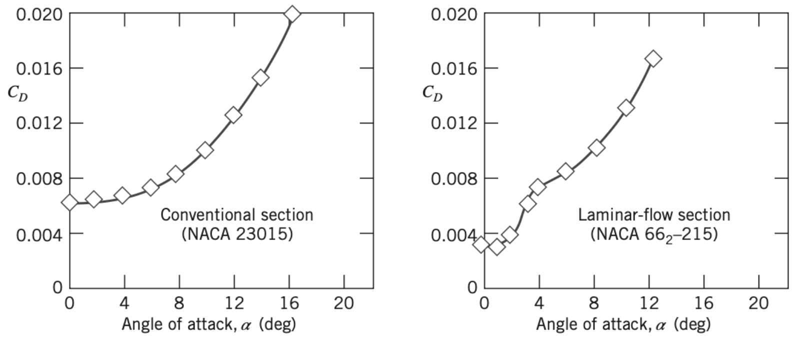


Figure 21.10: Drag Coefficient versus angle of attack for two airfoil sections of 15% thickness ratio at $\text{Re}_c = 9 \cdot 10^6$.

high C_L/C_D ratio is the goal for maximizing lifting ability whilst minimizing drag, at which laminar airfoils excel.

All airfoils are of finite span and therefore have less lift and more drag than their airfoil section data would indicate. This can be explained by the creation of vortices as depicted on Figure 21.11

Loss of lift and increase in drag caused by these finite-span effects are concentrated near the tip of the wing; hence, a short and stubby wing will experience these effects more severely than a long wing. We therefore expect the effects to correlate with the wing aspect ratio, defined as

$$\text{AR} \equiv \frac{b^2}{A_p}$$

where A_p is the planform area and b is the wingspan.

When the angle of attack of a finite wing is increased, the trailing vortices increase, leading to a loss of lift and increased drag. The effects of the finite aspect ratio can be characterized as a reduction Δa in the effective AoA. This is theoretically and experimentally determined to be approximately given by

$$\Delta\alpha \approx \frac{C_L}{\pi \text{AR}}.$$

The induced drag component of the drag coefficient is

$$\Delta C_D \approx C_L \Delta\alpha \approx \frac{C_L^2}{\pi \text{AR}}.$$

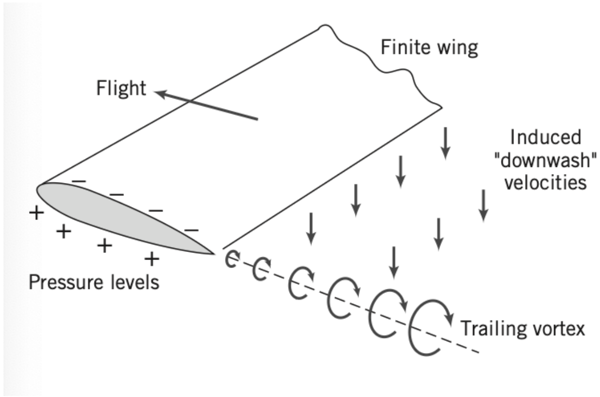


Figure 21.11: Schematic representation of the trailing vortex system of a finite wing.

This is depicted on Figure 21.12.

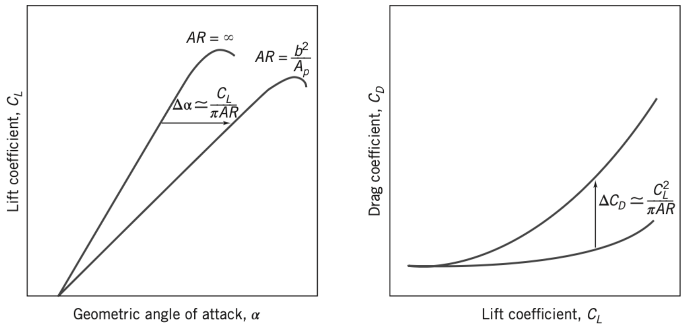


Figure 21.12: Effect of finite aspect ratio on lift and drag coefficients for a wing.

When written in terms of aspect ratio, the drag of a wing of finite span becomes

$$C_D = C_{D,\infty} + C_{D,i} = C_{D,\infty} + \frac{C_L^2}{\pi \mathrm{AR}}$$

where $C_{D,\infty}$ is the section drag coefficient at C_L, $C_{D,i}$ is the induced drag coefficient at C_L, and AR is the aspect ratio of the finite-span wing.

The effective aspect ratio of a wing can be improved by adding a winglet to the wing top. These are short, aerodynamically contoured wings set perpendicular to the wing at its tip. The winglet blocks the flow from crossing between the higher pressure region below the wing tip to the lower pressure region above the top. This reduces the formation of the trailing vortices which reduces drag on the aircraft.

Lecture 24: Intro to Turbomachinery Analysis and Pumps November 26, 2025

23 Fluid Machinery

Definition 7: Fluid Machines

A fluid machine is a device that either performs work on or extracts work from a fluid.

23.1 Classification of Fluid Machines

Fluid machines may broadly be classified as either *positive displacement* or *dynamic*. Dynamic fluid-handling devices that direct the flow using blades or vanes attached to a rotating member are termed *turbomachines*.

In positive-displacement machines, energy transfer is accomplished by volume changes that occur due to movement of the boundary in which the fluid is confined. This includes piston-cylinder arrangements, gear pumps and lobe pumps. These will only be reviewed shortly as the emphasis of the course is on dynamic machines.

Furthermore, one can also distinguish types of turbomachines based on the geometry of the flow path. In *radial-flow* machines, the flow path is (essentially) radial, with significant changes in radius from inlet to outlet. Such machines are sometimes called *centrifugal* machines. In *axial-flow* machines, the flow path is nearly parallel to the machine centerline, and the radius of the flow does not vary significantly. In *mixed-flow* machined the flow-path radius changes moderately.

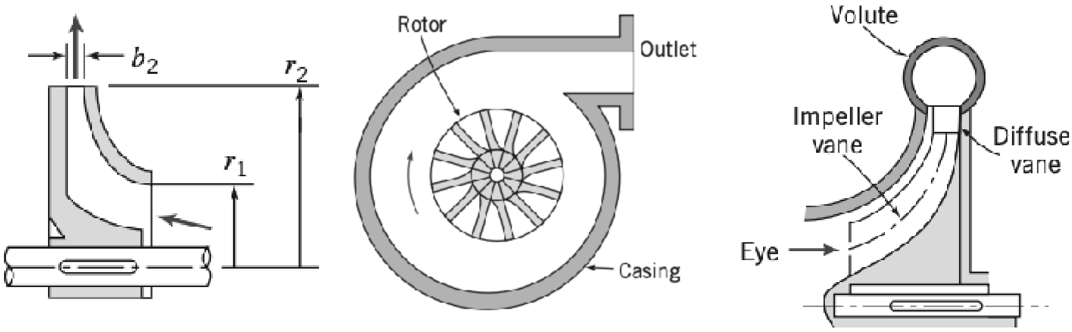
All work interactions in a turbomachine result from dynamic effects of the rotor on the fluid stream. I.e. the transfer of work between the fluid and the rotating machine will either increase or decrease the speed of flow. However, in conjunction with this kinetic energy transfer, machines that include external housing also involve either the conversion of pressure energy to kinetic energy or v.v.

23.1.1 Machines for Doing Work on a Fluid

Machines that add energy to a fluid by performing work on it are called *pumps*, when the flow is liquid or slurry and *fans*, *blowers*, or *compressors* for gas-handling units depending on pressure rise.

Pumps and compressors consist of a rotating wheel (called an *impeller* or *rotor*, depending on the type of machine) driven by an external power source to increase the kinetic energy of the flow, followed by an element to decelerate the flow, thereby increasing its pressure. This combination is known as a *stage*. A single pump or compressor might consist of several stages within a single housing depending on the required pressure rise.

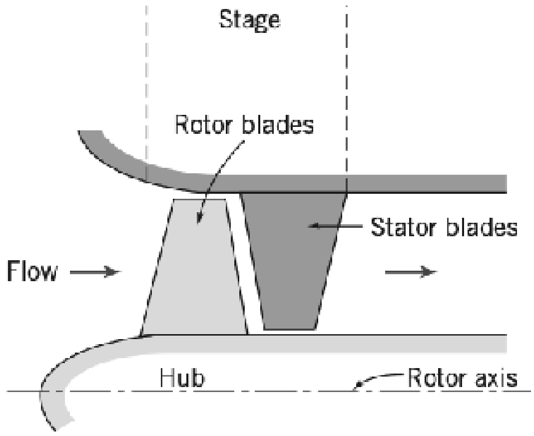
Three typical centrifugal machines are shown on Figure 23.1. Flow enters each machine nearly axially at small radius through the *eye* of the impeller (a), at radius r_1. Flow is turned and leaves through the impeller discharge at radius r_2, where the width is b_2. Ine increase in flow area reduces the fluid velocity and increases the pressure as shown on (b). (c) shows that the diffuser may have vanes to direct the flow between the impeller discharge and the *volute* (where the fluid is collected).



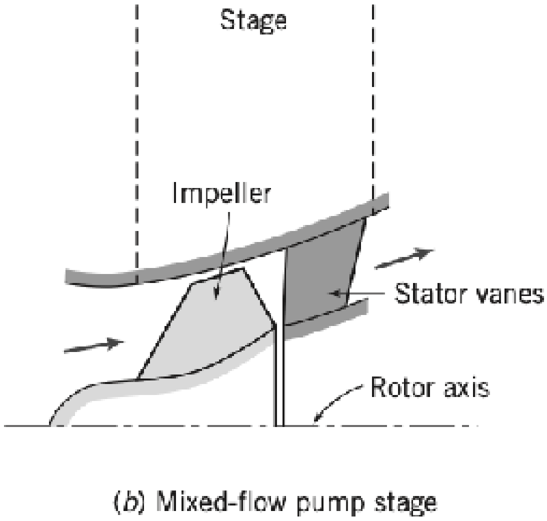
(a) Centrifugal pump (b) Centrifugal blower (c) Centrifugal compressor

Figure 23.1: Schematic of typical centrifugal-flow turbomachines.

Typical axial- and mixed-flow turbomachines are shown on Figure 23.2. (a) shows a typical axial-flow compressor stage. Here, the rotating element is called a rotor and flow diffusion happens in the stator. Flow enters nearly parallel to the rotor axis and maintains nearly the same radius through the stage. The mixed-flow pump on (b) shows the flow being turned outward moving to larger radius as it passes through the stage.



(a) Axial-flow compressor stage



(b) Mixed-flow pump stage

Figure 23.2: Schematic of typical axial- and mixed-flow turbomachines

23.1.2 Machines for Extracting Work From a Fluid

Machines that extract energy from a fluid in the form of work are called *turbines*. In *hydraulic turbines* the working fluid is water, so the flow is incompressible. In *gas turbines* and *steam turbines*, the density of the working fluid may change significantly. In a turbine, a stage normally consists of an element to accelerate the flow, converting some of its pressure energy to kinetic energy, followed by a *rotor*, *wheel*, or *runner* that extracts the kinetic energy from the flow via a set of *vanes*, *blades*, or *buckets* mounted on the wheel.

The two most general classifications of turbines are impulse and reaction turbines. *Impulse turbines* are driven by one or more high-speed free jets. The classic example of an impulse turbine is the waterwheel. In more modern forms of impulse turbines, the jet is accelerated in a nozzle external to the turbine wheel. If friction and gravity are neglected, neither the fluid pressure nor speed relative to the runner changes as the fluid passes over the turbine buckets. Thus, for an impulse turbine, the fluid acceleration and accompanying pressure drop take place in nozzles external to the blades, and work is extracted as a result of the large momentum change of the fluid.

In *reaction turbines*, part of the pressure change takes place externally and part takes place within the moving blades. This flow is turned to enter the runner in the proper direction as it passes through nozzles or stationary blades, called *guide vanes* or *wicket gates*. Additional fluid acceleration relative to the rotor occurs within the moving blades, so both the relative velocity and the pressure of the stream change across the runner. Because reaction turbines flow full of fluid, they generally can produce more power for a given size than impulse turbines.

23.2 Turbomachinery Analysis

23.2.1 The Angular-Momentum Principle: The Euler Turbomachine Equation

The angular-momentum principle was applied to control volumes Section 7.3, which resulted inSection 7.3.1:

$$\boldsymbol{r} \times \boldsymbol{F}_s + \int_{\text{CV}} \boldsymbol{r} \times \boldsymbol{g} \rho \, \mathrm{d}V + \boldsymbol{T}_{\text{shaft}} = \frac{\partial}{\partial t} \int_{\text{CV}} \boldsymbol{r} \times \boldsymbol{V} \rho \, \mathrm{d}V + \int_{\text{CV}} \boldsymbol{r} \times \boldsymbol{V} \rho \boldsymbol{V} \cdot \mathrm{d}\boldsymbol{A}.$$

Section 7.3.1 states that the moment of surface forces and body forces, plus the applied torque leads to a change in the angular momentum of the flow.

We will now simplify Section 7.3.1 for use in turbomachinery analysis. First it is convenient to choose a fixed control volume enclosing the rotor to evaluate shaft torque. Because we are looking at control volumes for which we expect large shaft torques, as a first approximation torques due to surface forces may be ignored. This means we are neglecting friction and torque generated by pressure changes. The body force may be neglected by symmetry. Then for steady flow, Section 7.3.1, reduces to:

$$\boldsymbol{T}_{\text{shaft}} = \int_{\text{CV}} \boldsymbol{r} \times \boldsymbol{V} \rho \boldsymbol{V} \cdot \mathrm{d}\boldsymbol{A}. \tag{49}$$

This states, that for a turbomachine with work *input*, the torque *required* causes a change in the angular momentum of the fluid. For a turbomachine with work *output*, the torque *produced* is due to the change in angular momentum of the fluid.

As depicted on Figure 23.3, we select a fixed control volume enclosing a generalized turbomachine rotor. The idealized velocity components are shown in the figure. The fluid enters the rotor at radial location r_1 with uniform absolute velocity, $\boldsymbol{V}_1$; the fluid leaves the rotor at radial location r_2 with uniform absolute velocity $\boldsymbol{V}_2$. For *uniform flow* into the rotor at Section 10.2 and out of the rotor at Section 10.2, Equation (49) becomes:

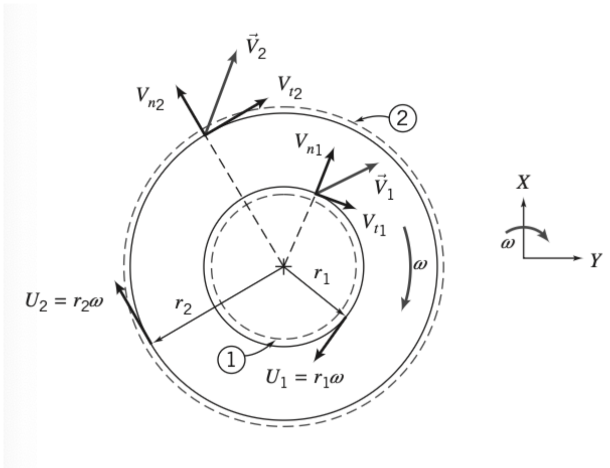


Figure 23.3: Finite control volume and absolute velocity components for analysis of angular momentum.

$$T_{\text{shaft}}\hat{k} = \left(r_2 V_{t_2} - r_1 V_{t_1}\right) \dot{m}\hat{k}.$$

For the product $\boldsymbol{r} \times \boldsymbol{V}$, the position vector is purely radial and only the tangential velocity component V_t counts. In scalar form:

$$T_{\text{shaft}} = \left(r_2 V_{t_2} - r_1 V_{t_1}\right) \dot{m}.$$

The assumptions behind this equation are *steady*, *frictionless flow*, *uniform flow* at inlet, and exit and *negligible pressure effects.*

The rate of work $\dot{W}_m$ done on a turbomachine rotor is given by the dot product of rotor angular velocity $\boldsymbol{\omega}$ and applied torque $\boldsymbol{T}_{\text{shaft}}$. Using the result from before, we obtain

$$\dot{W}_m = \boldsymbol{\omega} \cdot \boldsymbol{T}_{\text{shaft}} = \omega\hat{k} \cdot T_{\text{shaft}}\hat{k} = \omega\hat{k} \cdot \left(r_2 V_{t_2} - r_1 V_{t_1}\right) \dot{m}\hat{k}$$

or

$$\dot{W}_m = \omega T_{\text{shaft}} = \omega \left(r_2 V_{t_2} - r_1 V_{t_1}\right) \dot{m}.$$

We introduce $U = r\omega$, where U is the tangential speed of the rotor at radius r, we have

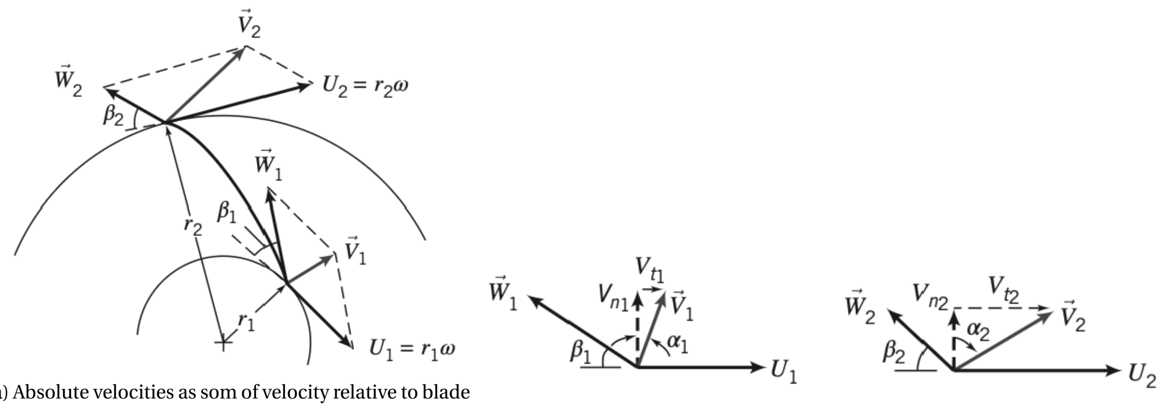
$$\dot{W}_m = \left(U_2 V_{t_2} - U_1 V_{t_1}\right) \dot{m}. \tag{50}$$

Dividing Equation (50) by $\dot{m}g$ we obtain a quantity with dimensions of length, which may be viewed as the theoretical head added to the flow:

$$H = \frac{\dot{W}_m}{\dot{m}g} = \frac{1}{g}\left(U_2 V_{t_2} - U_1 V_{t_1}\right).$$

23.2.2 Velocity Diagrams

The equations we derived in Section 23.2.1 suggest the importance of clearly defining the velocity components of the fluid and rotor at the inlet and outlet sections. For this purpose, it is useful to develop *velocity diagrams* for the inlet and outlet flows. Figure 23.4 shows the velocity diagrams and introduced the notation for blade and flow angles. The important notation to remember is that the variable V is typically used to indicate absolute velocity, while the variable W is used to indicate flow velocity relative to the rotating blade.



(a) Absolute velocities as som of velocity relative to blade and rotor velocity

(b) Velocity components at inlet

(c) Velocity components at outlet

Figure 23.4: Geometry and notation used to develop velocity diagrams for radial-flow machines.

Machines are designed in a manner, such that at *design condition* the fluid moves smoothly through the blades. In the idealized situation at the *design speed*, sometimes called *shockless entry*, flow relative to the rotor is assumed to enter and leave tangent to the blade profile at each section. At speeds other than the design speed, the fluid may impact the blades at inlet, exit at an angle relative to the blade, or may have significant flow separation, leading to inefficiency. Figure 23.4 is representative of a typical radial flow machine. We assume that the fluid is moving without major flow disturbances through the machine, as shown in Figure 23.4a, with blade inlet and exit angles β_1 and β_2, respectively, relative to the circumferential direction. In general β_1 and β_2 may be any angle from $0°$ to $180°$.

The runner speed at the inlet is $U_1 = r_1\omega$, and therefore it is specified by the impeller geometry and the operating speed. The absolute fluid velocity is the vector sum of the impeller velocity and the flow velocity relative to the blade. The absolute inlet velocity may be determined graphically, as shown in Figure 23.4b. The angle of the absolute fluid velocity α_1 is measured from the direction normal to the flow area. For a given machine, α_1 and α_2 will vary with flow rate Q and rotor speed ω. The tangential component of the absolute velocity V_{t_1} and the component normal to the flow area V_{n_1} are also shown on Figure 23.4b. At each section the normal component of the absolute velocity V_n and the normal component relative to the velocity of the blade W_n are equal because the blade has no normal velocity.

The velocity diagram is constructed similarly at the outlet section. The runner speed at the outlet is $U_2 = r_2\omega$, which again is known from the geometry and operating speed of the turbomachine. The relative flow is assumed to leave the impeller tangent to the blades as shown on Figure 23.4c.

For a centrifugal pump or reaction turbine, the velocity relative to the blade generally changes in magnitude from the inlet to outlet. The continuity equation must be applied to determine the normal component of velocity at each section. The normal component together with the flow outlet blade angle is sufficient to establish the velocity relative to the blade impeller outlet for a radial-flow machine. The velocity diagram is completed by the vector addition of the velocity relative to the blade and the wheel velocity as depicted on Figure 23.4c.

Due to the idealized assumptions the analytical result will often represent a sort of upper limit of the performance of actual machines.

23.2.3 Performance–Hydraulic Power

The torque and power predicted by applying the angular momentum equation to a turbomachine rotor are idealized values. In practice, rotor power and the rate of change of fluid energy are not equal. Losses are caused by viscous effects, non-uniform flow, mismatched flow direction and blade angle, and inefficiencies in the diffuser. Energy dissipation occurs in seals and bearings and in fluid friction between the rotor and housing of the machine (so-called *windage* losses). Because of these losses, in a pump, the actual power delivered to the fluid is less than predicted by the angular-momentum equation. In the case of a turbine, the actual power delivered to the shaft is less than the power given up by the fluid stream.

We define the power, head and efficiency of a turbomachine based on whether the machine does work on the fluid or extracts work from the fluid. For a pump, the *hydraulic power* is given by the rate of mechanical energy input to the fluid,

$$\dot{W}_h = \rho Q g H_p$$

where

$$H_p = \left(\frac{p}{\rho g} + \frac{\overline{V}^2}{2g} + z\right)_{\text{discharge}} - \left(\frac{p}{\rho g} + \frac{\overline{V}^2}{2g} + z\right)_{\text{suction}}.$$

The mechanical input power needed to drive the pump is greater then that to produce the head rise due to inefficiencies. We define the *pump efficiency as*

$$\eta_p \equiv \frac{\dot{W}_h}{\dot{W}_m} = \frac{\rho Q g H_p}{\omega T}.$$

For a hydraulic turbine, the *hydraulic power* is defined as the rate of mechanical energy removal from the flowing fluid stream,

$$\dot{W}_h = \rho Q g H_t$$

where

$$H_p = \left(\frac{p}{\rho g} + \frac{\overline{V}^2}{2g} + z\right)_{\text{inlet}} - \left(\frac{p}{\rho g} + \frac{\overline{V}^2}{2g} + z\right)_{\text{outlet}}.$$

The mechanical power output obtained form the turbine is related to the hydraulic power by defining *turbine efficiency* as

$$\eta_t \equiv \frac{\dot{W}_m}{\dot{W}_h} = \frac{\omega T}{\rho Q g H_t}.$$

Therefore, to obtain maximum power from a hydraulic turbine, one must minimize the mechanical energy in the flow leaving the turbine. This is accomplished by making the outlet pressure, flow speed, and elevation as small as practical.

23.3 Pumps, Fans, and Blowers

23.3.1 Application of Euler Turbomachine Equation to Centrifugal Pumps

With $V_{t_1} = 0$, the increase in head is given by

$$H = \frac{U_2 V_{t_2}}{g}.$$

From the exit velocity diagram of Figure 23.4c,

$$V_{t_2} = U_2 - W_2 \cos\beta_2 = U_2 - \frac{V_{n_2}}{\sin\beta_2}\cos\beta_2 = U_2 - V_{n_2}\cot\beta_2.$$

Then

$$H = \frac{U_2^2 - U_2 V_{n_2}\cot\beta_2}{g}.$$

For an impeller of width w, the volume flow rate is

$$Q = \pi D_2 w V_{n_2}. \tag{51}$$

To express the increase in head in terms of volume flow rate, we substitute for V_{n_2} in terms of Q from Equation (51). Thus

$$H = \frac{U_2^2}{g} - \frac{U_2\cot\beta_2}{\pi D_2 w g}Q. \tag{52}$$

This is an equation of the form

$$H = C_1 - C_2 G$$

where constants C_1 and C_2 are functions of *machine geometry* and *speed*.

Therefore, Equation (52) predicts a linear variation of head H with volume flow rate Q. Constant $C_1 = U_2^2/g$ represents the ideal head developed by the pump for zero flow rate and is called the *shutoff head*. The linear relation is an idealized model and actual devices may only approximate linear variation.

For radial outlet vanes, $\beta_2 = 90°$ and $C_2 = 0$. The tangential component of the absolute velocity at the outlet is equal to the wheel speed and is independent of flow rate. From Equation (52) the ideal head is independent of flow rate. The characteristic $H - Q$ curve is shown on Figure 23.5.

If the vanes are forward curved, then $\beta_2 > 90°$ and $C_2 < 0$. The tangential component of the absolute fluid velocity at the outlet is greater than the wheel speed, and it increases as the flow rate increases.

23.3.2 Performance Characteristics

To specify fluid machines for flow systems, the designer must know the pressure rise (or head), torque, power requirement, and efficiency of a machine. For a given machine, each of these characteristics is a function of flow rate and for similar machines depend on size and operating speed.

On Figure 23.6 idealized and actual head-flow curves are plotted. This shows that the head at any flow rate in the real machine may be significantly lower than predicted by the idealized analysis. Some of the causes are that at very low flow rate, some fluid recirculates in the impeller, friction loss and leakage loss both increase with flow rate, and "shock loss" results from a mismatch between the direction of the relative velocity and the tangent to the impeller blade at the inlet.

The curve shown on Figure 23.6 is measured at constant (design) speed with a single impeller diameter. It is common practice to vary pump capacity by changing the impeller size in a given casing.

23.3.3 Pump Selection in Fluid Systems

Definition 8: Fluid System

We define a *fluid system* as the combination of a fluid machine and a network of pipes or channels that convey fluid.

A typical pump produces a smaller head at higher pressure as the flow rate is increased. In contrast, the head

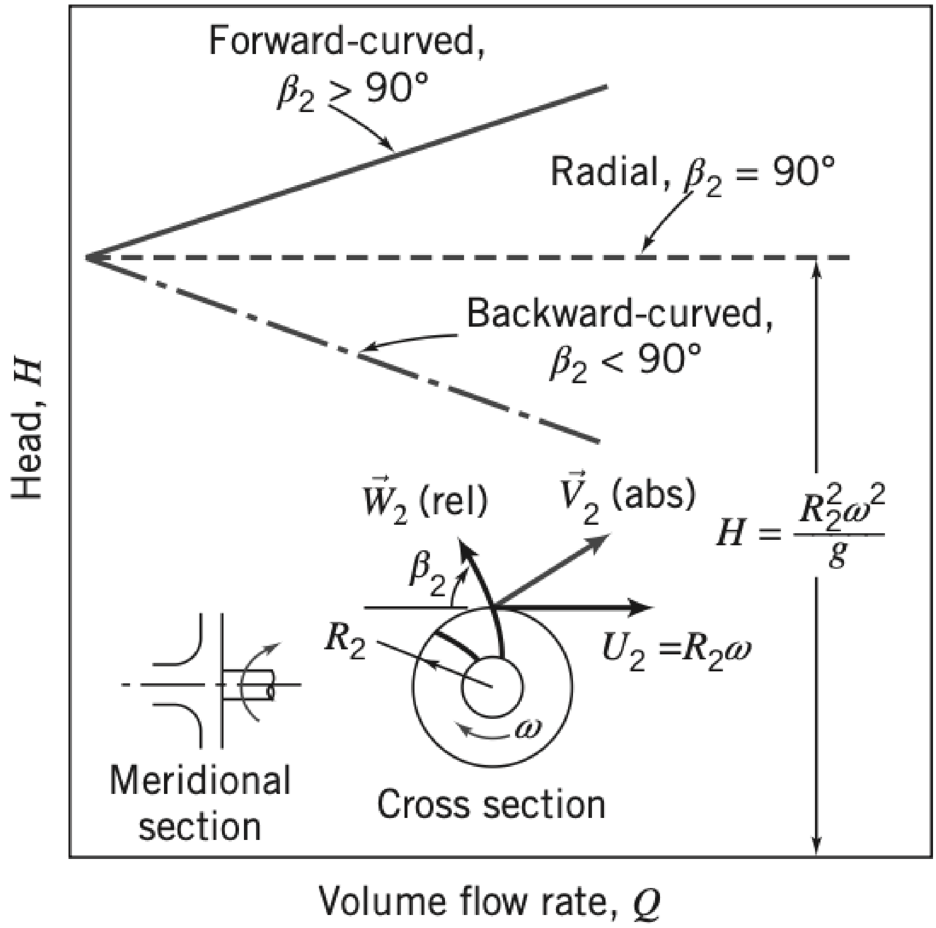


Figure 23.5: Idealized relationship between head and volume flow rate for centrifugal pump with forward-curved, radial, and backward-curved impeller blades.

required to maintain flow in a pipe system increases with the flow rate due to increased losses. Therefore a pump system will operate at the flow rate at which the pump head rise and required system head match. This is termed the operating point.

The required head for a system with no static lifts starts at zero flow and head and increases with flow. In this case, the total head required is the sum of major and minor losses,

$$h_{l_t} = \sum h_l + \sum h_m = \sum f \frac{L}{D}\frac{\overline{V}^2}{2} + \sum K \frac{\overline{V}^2}{2}.$$

For turbulent flow, the friction factors are nearly constant with flow and the minor loss coefficients K are also constant. Hence $h_{l_t} \sim \overline{V}^2 \sim Q^2$ so that the system curve is approximately parabolic. This means that the system curve with pure friction becomes steeper as flow rate increases.

Whether the resulting system curve is *steep* or *flat* depends on the relative importance of friction and gravity. Friction drop may be relatively unimportant in the water supply to a high-rise building and gravity lift may be negligible in an air-handling system for a one-story building.

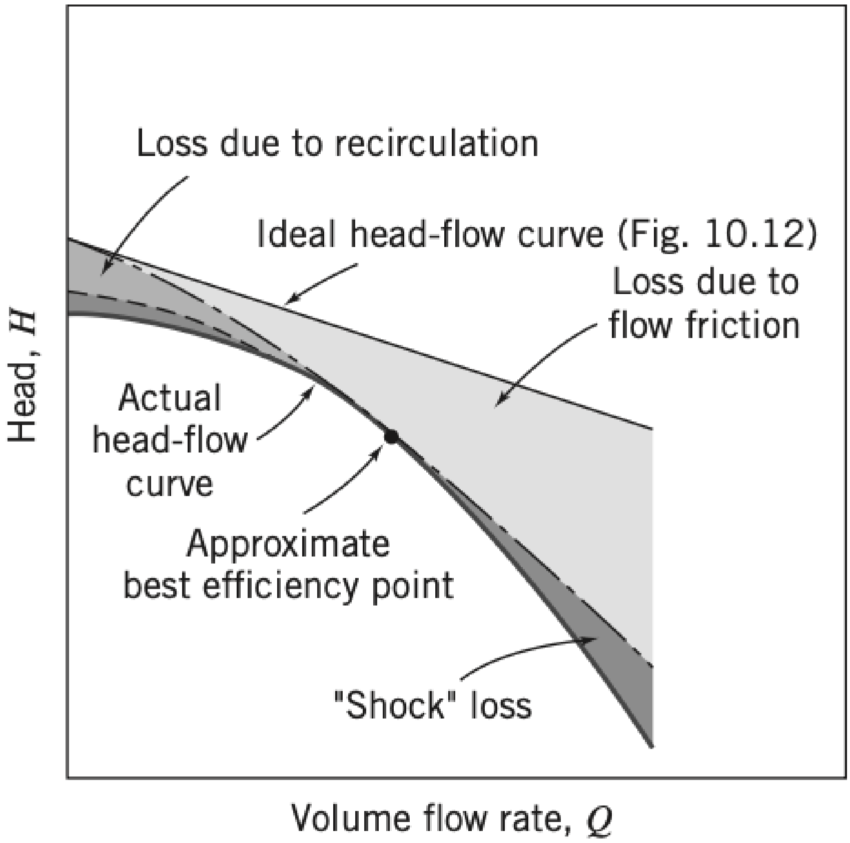


Figure 23.6: Comparison of ideal and actual head-flow curves for a centrifugal pump with backward-curved impeller blades

Lecture 25: Introduction to Compressible Flow December 1, 2025

25 Introduction to Compressible Flow

Any flow in which density variation is not negligible is termed *compressible*. In general fluids are never compressible and gasses are compressible, when the Mach number $M \equiv V/c > 0{,}3$ where c is the local speed of sound and V is the velocity. Flows with speed less than the speed of sound $M < 1$ are called subsonic. Flows with speeds greater than the speed of sound are *supersonic*. Flows with $0{,}9 < M < 1{,}2$ are talled *transonic* and flows with $5 < M$ are called *hypersonic*.

25.1 Propagation of Sound Waves

25.1.1 Speed of Sound

The speed of sound c is an important property in compressible fluid flow. The Mach number M is defined as

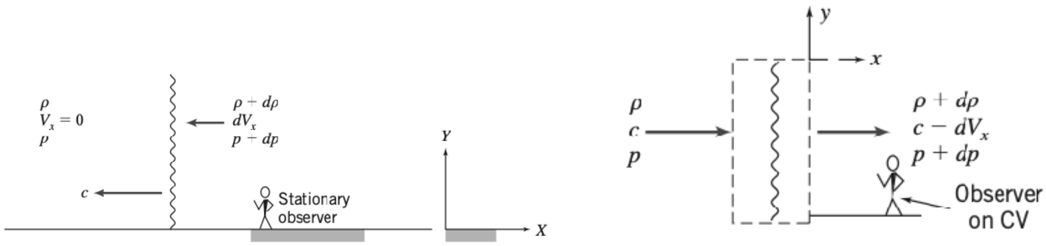
$$M \equiv \frac{V}{c}$$

where V is the speed of the fluid.

We consider the propagation of a sound wave of infinitesimal strength into an undisturbed medium, shown on Figure 25.1a. We want to relate the speed of wave propagation c to fluid property changes across the wave. We denote the pressure and density in the undisturbed medium by p and ρ respectively. Passage of the wave will cause these to undergo infinitesimal changes to become $p + \mathrm{d}p$ and $\rho + \mathrm{d}\rho$. Since the wave propagates into a stationary fluid, the velocity ahead of the wave V_x is zero. The magnitude of the velocity behind the wave

$V_x + \mathrm{d}V_x$ will thus simply be $\mathrm{d}V_x$.

The flow of Figure 25.1a appears unsteady to a stationary observer, viewing the wave motion from a fixed point. However, the flow appears steady to an observer located *on* an inertial control volume moving with a segment of the wave as shown on Figure 25.1b. The velocity approaching the control volume is then c and the velocity leaving is $c - \mathrm{d}V_x$.



(a) Propagating wave

(b) Inertial control volume moving with wave, velocity c

Figure 25.1: Propagating sound wave showing control volume chosen for analysis.

Assuming steady flow and uniform flow at each section the continuity equation is

$$\int_{\mathrm{CS}} \rho \boldsymbol{V} \cdot \mathrm{d}\boldsymbol{A} = 0.$$

which can also be written as

$$\mathrm{d}V_x = \frac{c}{\rho}\mathrm{d}\rho.$$

Furthermore, assuming negligible body forces the momentum equation becomes:

$$F_{S_x} = \int_{\mathrm{CS}} V_x \rho \boldsymbol{V} \cdot \mathrm{d}\boldsymbol{A}.$$

The only surface acting in the x direction on the control volume of Figure 25.1b are due to pressure. The upper and lower areas have zero friction as the areas are infinitesimal. This leaves us with:

$$\mathrm{d}V_x = \frac{1}{\rho c}\mathrm{d}p.$$

Combining this with the previous expression for $\mathrm{d}V_x$, we obtain

$$\mathrm{d}V_x = \frac{c}{\rho}\mathrm{d}\rho = \frac{1}{\rho c}\mathrm{d}p$$

from which

$$\mathrm{d}p = c^2 \mathrm{d}\rho$$

or

$$c^2 = \frac{\mathrm{d}p}{\mathrm{d}\rho}.$$

This indicates that the speed of sound depend on how the pressure and density of the medium are related. Sound waves are infinitesimal and happen quickly, hence they propagate isentropically. Therefore, if we express p as a function of density and entropy $p = p(\rho, s)$, then

$$\mathrm{d}p = \left(\frac{\partial p}{\partial s}\right)_s \mathrm{d}\rho + \left(\frac{\partial p}{\partial s}\right)_\rho \mathrm{d}s = \left(\frac{\partial p}{\partial \rho}\right)_s \mathrm{d}\rho.$$

Therefore:

$$c^2 = \frac{\mathrm{d}p}{\mathrm{d}\rho} = \left.\frac{\partial p}{\partial \rho}\right)_s$$

and

$$c = \sqrt{\left.\frac{\partial p}{\partial \rho}\right)_s}. \tag{53}$$

We will now apply Equation (53) to solids, liquids, and gases. For solids and liquids data are usually available on the bulk modulus E_v, which is a measure of how a pressure change affects a relative density change:

$$E_v = \frac{\mathrm{d}p}{\frac{\mathrm{d}\rho}{\rho}} = \rho \frac{\mathrm{d}p}{\mathrm{d}\rho}.$$

For these media

$$c = \sqrt{\frac{e_v}{\rho}}. \tag{54}$$

For an ideal gas, the pressure and density in isentropic flow are related as

$$\frac{p}{\rho^k} = \text{constant}.$$

I.e.:

$$\frac{\mathrm{d}p}{p} - k\frac{\mathrm{d}\rho}{\rho} = 0$$
$$\left.\frac{\partial p}{\partial \rho}\right)_s = k\frac{p}{\rho}.$$

But $p/\rho = RT$, so

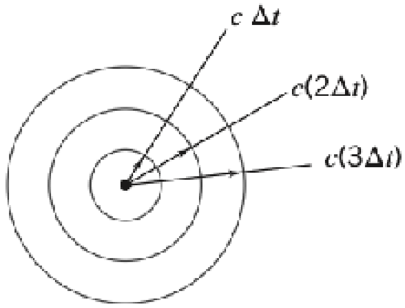
$$c = \sqrt{kRT} \tag{55}$$

for an ideal gas.

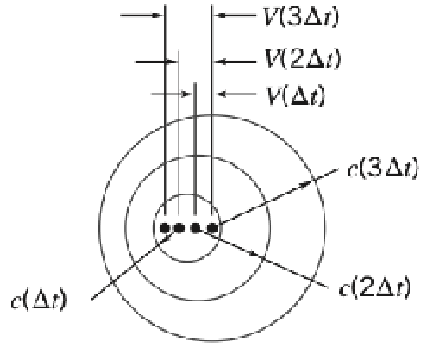
25.1.2 Supersonic Motion – The Mach Cone

We consider a point source of sound that emits a pusle every Δt seconds. Each pulse expands outwards from its origination point at the speed of sound c, so at any instant t the pulse will be a sphere of radius ct centered at the pulse's origination point. We want to investigate what happens if the point source itself is moving. There are four possibilities, each shown on Figure 25.2.

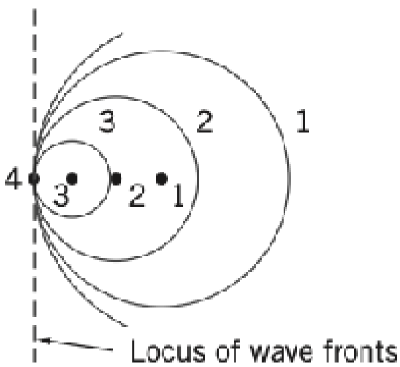
1. $V = 0$. The point is *stationary*. Figure 25.2a shows the conditions for the scenario after $3\Delta t$ seconds. The first pulse has expanded to a sphere of radius $c(3\Delta t)$, the second to a sphere of radius $c(2\Delta t)$, and the third to a sphere of radius $c(\Delta t)$; a new pulse is about to be emitted. The pulses therefore constitute a set of ever-expanding concentric spheres.
2. $0 < V < c$. The point source moves to the left at *subsonic* speed. Figure 25.2b shows the conditions for the scenario after $3\Delta t$ seconds. The source is shown at times $t = 0, \Delta t, 2\Delta t$, and $3\Delta t$. The first pulse har expanded to a sphere radius $c(3\Delta t)$ centered where the source was originally, the second to a sphere of radius $c(2\Delta t)$ centered where the source was at time t and so on. Again, the pulses constitute a set of ever-expanding spheres, except now they are not concentric. The pulses are all propagating at constant speed c. Note that an observer ahead of the source will hear the pulses at a higher frequency than an observer behind the source. This is known as the Doppler effect. Also, note that an observer ahead of the sound will hear the source *before* the source itself reaches the observer.
3. $V = c$. The point source moves to the left at *sonic* speed. Figure 25.2c shows conditions after $3\Delta t$ seconds.



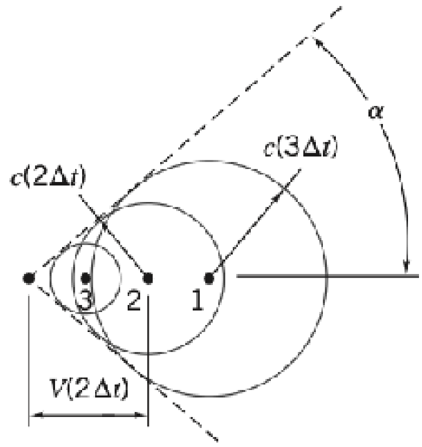
(a) $V = 0$: Stationary source



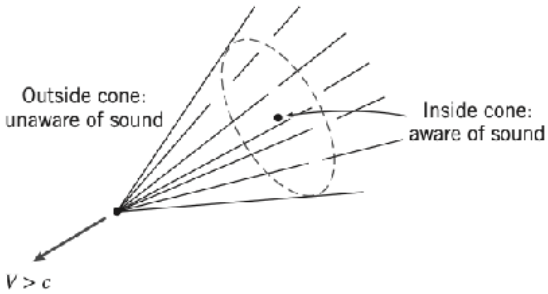
(b) $V < c$: Doppler shift



(c) $V = c$



(d) $V > c$: Supersonic motion



(e) $M > 1$: The Mach cone

Figure 25.2: Propagation of sound waves from a moving source: The Mach cone.

The source is shown at times $t = 0$, Δt, $2\Delta t$, and $3\Delta t$. The first pulse has expanded to sphere 1 of radius $c(3\Delta t)$ centered at point 1, the second to sphere 2 of radius $c(2\Delta t)$ centered at point 2 and so on. We can see once more that the pulses constitute a set of ever-expanding spheres, except now they are tangent to one another on the left. The pulses are all expanding at constant speed c, but the source is moving at speed c, with the result that the source and all its pulses are travelling together to the left. Here we note that an observer who is ahead of the source will *not* hear the pulses before the source reaches the observer. Secondly, over time an unlimited number of pulses will accumulate at the front of the source lading to a sound wave of unlimited amplitude.

4. $V > c$. The point source moves to the left at *supersonic* speed. Figure 25.2d shows the conditions after $3\Delta t$ seconds. We can see once more that the pulses constitute a set of ever-expanding spheres, except now the source is moving so fast it moves ahead of each sphere that it generates. For supersonic motion, the spheres generate what is called a *Mach cone* tangent to each sphere. In this case, an observer who is ahead of the source will not hear the pulses until after the source passes the observer. The region inside the cone

is called the *zone of action* and that outside the cone the *zone of silence* as shown in Figure 25.2e. From geometry, we see from Figure 25.2d, that

$$\sin\alpha = \frac{c}{V} = \frac{1}{M}.$$

or

$$\alpha = \sin^{-1}\frac{1}{M}. \tag{56}$$

25.2 Reference State: Local Isentropic Stagnation Properties

In our study of compressible flow, we will discover that, in general, *all* properties (p, T, ρ, u, s, V) may be changing as the flow proceeds. We need to obtain reference conditions that we can use to relate conditions in a flow from point to point. For any flow, a reference condition is obtained when the fluid is brought to rest either in reality or conceptually. We will call this the *stagnation condition*, and the property values $(p_0, T_0, \rho_0, u_0, h_0, s_0)$ at this state the *stagnation properties.* The stagnation state is defined by an isentropic process, in which there will is no friction, no heat transfer, and no "violent" events. Hence, the properties we obtain will be the *local isentropic stagnation properties* and, each point in the flow will have its own, or local, isentropic stagnation properties. This is depicted on Figure 25.3 showing a flow from state 1 to some new state 2. The local isentropic stagnation properties for each state is obtained by isentropically bringing the fluid to rest, are also shown. Hence, $s_{0_1} = s_1$ and $s_{0_2} = s_2$. If the flow is isentropic, $s_1 = s_2 = s_{0_1} = s_{0_2}$, so the stagnation states are identical; if it is *not* isentropic, then $s_{0_1} = s_{0_2}$.

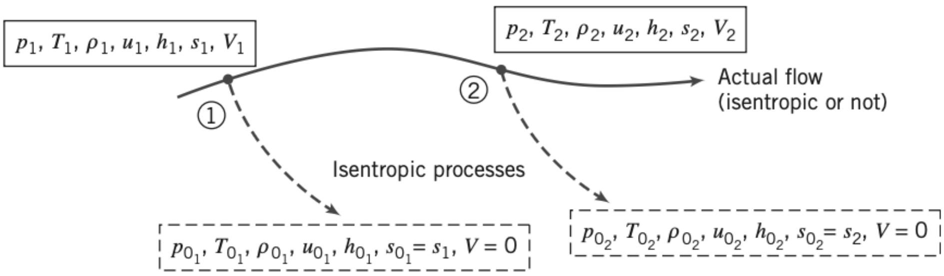


Figure 25.3: Local isentropic stagnation properties

We recall the Bernoulli equation

$$\frac{p}{\rho} + \frac{V^2}{2} + gz = \text{constant}$$

valid for a steady, incompressible, frictionless flow along a streamline. This is valid for an incompressible isentropic process because it is reversible (frictionless and steady) and adiabatic. It was previously shown that the Bernoulli equation leads to

$$p_0 = p + \frac{1}{2}\rho V^2.$$

The gravity term drops out because we assume the reference state is at the same elevation as the actual state, and in any event in external flows it is usually much smaller than the other terms.

25.2.1 Local Isentropic Stagnation Propertied for the Flow of an Ideal Gas

For a compressible flow we can derive the isentropic stagnation relations by applying the mass conservation and momentum equations to a differential control volume, and then integrating. For the process shown schematically on Figure 25.3, we can depict the process from state 1 to the corresponding stagnation state by imagining the control volume shown in Figure 25.4.

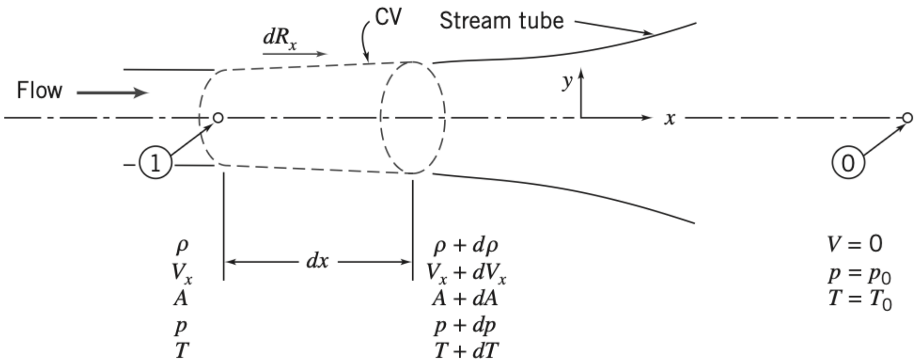


Figure 25.4: Compressible flow in an infinitesimal stream tube.

For steady flow and uniform flow at each section the continuity equation reduces to:

$$\begin{aligned}\int_{\mathrm{CS}} \rho \boldsymbol{V} \cdot \mathrm{d}\boldsymbol{A} &= 0 \\ \left(-\rho V_x A\right) + \left(\left(\rho + \mathrm{d}\rho\right)\left(V_x + \mathrm{d}V_x\right)\left(A + \mathrm{d}A\right)\right) &= 0 \\ \left(\rho + \mathrm{d}\rho\right)\left(V_x + \mathrm{d}V_x\right)\left(A + \mathrm{d}A\right) &= \rho V_x A. \end{aligned} \tag{57}$$

For negligible body forces and frictionless flow the momentum equation reduces to

$$F_{S_x} = \int_{\mathrm{CS}} V_x \rho \boldsymbol{V} \cdot \mathrm{d}\boldsymbol{A}.$$

The surface forces acting on the infinitesimal control volume are

$$F_{S_x} = \mathrm{d}R_x + pA - \left(p + \mathrm{d}p\right)\left(A + \mathrm{d}A\right).$$

The force $\mathrm{d}R_x$ is applied along the stream boundary as shown in Figure 25.4, where the average pressure is $p + \mathrm{d}p/2$, and the area component in the x direction is $\mathrm{d}A$. There is no friction, thus,

$$F_{S_x} = \left(p + \frac{\mathrm{d}p}{2}\right)\mathrm{d}A + pA - \left(p + \mathrm{d}p\right)\left(A + \mathrm{d}A\right)$$

or

$$F_{S_x} = -\mathrm{d}pA.$$

Substituting this result into the momentum equation gives

$$-\mathrm{d}pA = V_x\left(-\rho V_x A\right) + \left(V_x + \mathrm{d}V_x\right)\left(\left(\rho + \mathrm{d}\rho\right)\left(V_x + \mathrm{d}V_x\right)\left(A + \mathrm{d}A\right)\right)$$

which may be simplified using Equation (57) to yield

$$-\mathrm{d}pA = \left(-V_x + V_x + \mathrm{d}V_x\right)\left(\rho V_x A\right).$$

Finally,

$$\mathrm{d}p = -\rho V_x \mathrm{d}V_x = -\rho\,\mathrm{d}\left(\frac{V_x^2}{2}\right)$$

or

$$\frac{\mathrm{d}p}{\rho} + \mathrm{d}\left(\frac{V_x^2}{2}\right) = 0.$$

This is a relation among properties during the deceleration process. In developing this relation, we have specified a frictionless deceleration process. Before we can integrate between the initial state and final stagnation state, we must specify the relation that exists between pressure p and density ρ along the process path.

Since the deceleration process is isentropic, then p and ρ for an ideal gas are related by the expression

$$\frac{p}{\rho^k} = \text{constant}.$$

From $p/\rho^k = \text{constant} = C$, we get

$$p = C\rho^k \qquad \text{and} \qquad \rho = p^{\frac{1}{k}} C^{-\frac{1}{k}}.$$

Therefore:

$$-\mathrm{d}\left(\frac{V^2}{2}\right) = \frac{\mathrm{d}p}{\rho} = p^{-\frac{1}{k}} C^{\frac{1}{k}}\,\mathrm{d}p.$$

We integrate this equation between the initial state and the corresponding stagnation state

$$-\int_V^0 \mathrm{d}\left(\frac{V^2}{2}\right) = C^{\frac{1}{k}} \int_p^{p_0} p^{-\frac{1}{k}}\,\mathrm{d}p$$

to obtain

$$\frac{V^2}{2} = C^{\frac{1}{k}} \frac{k}{k-1} p^{\frac{k-1}{k}} \left(\left(\frac{p_0}{p}\right)^{\frac{k-1}{k}} - 1\right).$$

Since $C^{1/k} = p^{1/k}/\rho$,

$$\frac{V^2}{2} = \frac{k}{k-1}\frac{p}{\rho}\left(\left(\frac{p_0}{p}\right)^{\frac{k-1}{k}} - 1\right).$$

As we seek an expression for stagnation pressure, we rewrite as:

$$\left(\frac{p_0}{p}\right)^{\frac{k-1}{k}} = 1 + \frac{k-1}{k}\frac{\rho}{p}\frac{V^2}{2}$$

and

$$\frac{p_0}{p} = \left(1 + \frac{k-1}{k}\frac{\rho V^2}{2p}\right)^{\frac{k}{k-1}}.$$

For an ideal gas, $p = \rho RT$, and hence

$$\frac{p_0}{p} = \left(1 + \frac{k-1}{2}\frac{V^2}{kRT}\right)^{\frac{k}{k-1}}.$$

Also, for an ideal gas, $p = \rho RT$, and hence

$$\frac{p_0}{p} = \left(1 + \frac{k-1}{2}\frac{V^2}{kRT}\right)^{\frac{k}{k-1}}.$$

Also, for an ideal gas the sonic speed is $c = \sqrt{kRT}$, and thus

$$\frac{p_0}{p} = \left(1 + \frac{k-1}{2}\frac{V^2}{c^2}\right)^{\frac{k}{k-1}}$$

$$\frac{p_0}{p} = \left(1 + \frac{k-1}{2}M^2\right)^{\frac{k}{k-1}}.$$

This equation enables us to calculate the local isentropic stagnation pressure at any point in a flow field of an ideal gas, provided we know the static pressure and Mach number at that point.

Hence, the equations for determining the local isentropic stagnation properties of an ideal gas are

$$\frac{p_0}{p} = \left(1 + \frac{k-1}{2}M^2\right)^{\frac{k}{k-1}} \tag{58}$$
$$\frac{T_0}{T} = 1 + \frac{k-1}{2}M^2 \tag{59}$$
$$\frac{\rho_0}{\rho} = \left(1 + \frac{k-1}{2}M^2\right)^{\frac{1}{k-1}}. \tag{60}$$

25.3 Reference State: Critical Conditions

Stagnation conditions are useful as reference conditions for thermodynamic properties. A useful reference value for velocity is the *critical speed*, which is the speed V we attain when a flow is either accelerated or decelerated isenotpisally until $M = 1$. Even if there is no point in a given flow field there the Mach number is equal to unity, such a hypothetical condition still is useful as a reference condition.

Using asterisks to denote conditions at $M = 1$, then by definition,

$$V^* \equiv c^*.$$

At critical conditions the stagnation equations become

$$\frac{p_0}{p^*} = \left(\frac{k+1}{2}\right)^{\frac{k}{k-1}} \tag{61}$$
$$\frac{T_0}{T^*} = \frac{k+1}{2} \tag{62}$$
$$\frac{\rho_0}{\rho^*} = \left(\frac{k+1}{2}\right)^{\frac{1}{k-1}}. \tag{63}$$

The critical speed may be written in terms of either critical temperature T^* or isentropic stagnation temperature T_0.

For an ideal gas $c^* = \sqrt{kRT^*}$, and thus $V^* = \sqrt{kRT^*}$. Since:

$$T^* = \frac{2}{k+1}T_0$$

we have

$$V^* = c^* = \sqrt{\frac{2k}{k+1}RT_0}.$$

25.4 Isentropic Flow of an Ideal Gas: Area Variation

In the absence of heat transfer, friction, and shocks a flow will be reversible and adiabatic. A general result is

$$\dot{m}(s_2 - s_1) = \int_{\mathrm{CS}} \frac{1}{T}\left(\frac{\dot{Q}}{A}\right) \mathrm{d}A = 0$$

or

$$\Delta s = s_2 - s_1 = 0$$

so such a flow is *isentropic*.

We also have that

$$T_1 p_1^{\frac{1-k}{k}} = T_2 p_2^{\frac{1-k}{k}} = T p^{\frac{1-k}{k}} = \text{constant}$$

or its equivalent
$$\frac{p_1}{\rho_1^k} = \frac{p_2}{\rho_2^k} = \frac{p}{\rho^k} = \text{constant.}$$

This leads to the set of equations:

$$\rho_1 V_1 A_1 = \rho_2 V_2 A_2 = \rho V A = \dot{m} = \text{constant} \tag{64}$$
$$R_x + p_1 A_1 - p_2 A_2 = \dot{m} V_2 - \dot{m} V_1 \tag{65}$$
$$h_{0_1} = h_1 + \frac{V_1^2}{2} = h_2 + \frac{V_2^2}{2} = h_{0_2} = h_0 \tag{66}$$
$$s_2 = s_1 = s \tag{67}$$
$$p = \rho R T \tag{68}$$
$$\Delta h = h_2 - h_1 = c_p \Delta t = c_p (T_2 - T_1) \tag{69}$$
$$\frac{p_1}{\rho_1^k} = \frac{p_2}{\rho_2^k} = \frac{p}{\rho^k} = \text{constant.} \tag{70}$$

Equation (64) could be used to analyze isentropic flow in a channel of varying area. For example if we know the conditions at section 1 we could use these equations to find conditions at some new section 2, where the area is A_2.

25.4.1 Reference Stagnation and Critical Conditions for Isentropic Flow of an Ideal Gas

Using Equation (64) to analyze one-dimensional isentropic flow of an ideal gas, would be somewhat tedious. Instead, as the flow is *isentropic*, we can find the result in a simpler way Instead of using Equation (64) to compute, e.g. the properties at state 2 from those at state 1, we can use state 1 to determine two reference states, and then use these to obtain properties at state 2. We need two reference states because the reference stagnation state does not provide area information.

We previously found

$$\frac{p_0}{p} = \left(1 + \frac{k-1}{2} M^2\right)^{\frac{k}{k-1}} \tag{71}$$
$$\frac{T_0}{T} = 1 + \frac{k-1}{2} M^2 \tag{72}$$
$$\frac{\rho_0}{\rho} = \left(1 + \frac{k-1}{2} M^2\right)^{\frac{1}{k-1}}. \tag{73}$$

We note that the *stagnation conditions* are *constant throughout the isentropic flow*. The critical conditions $M = 1$ are related to the stagnation conditions previously introduced:

$$\frac{p_0}{p^*} = \left(\frac{k+1}{2}\right)^{\frac{k}{k-1}} \tag{74}$$
$$\frac{T_0}{T^*} = \frac{k+1}{2} \tag{75}$$
$$\frac{\rho_0}{\rho} = \left(\frac{k+1}{2}\right)^{\frac{1}{k-1}} \tag{76}$$
$$V^* = c^* = \sqrt{\frac{2k}{k+1} R T_0}. \tag{77}$$

Although a particular flow may never attain sonic conditions, we still find the critical conditions useful as a reference state.

We can obtain a relation between areas A and A^* from the continuity equations and get:

$$\frac{p_0}{p} = \left(1 + \frac{k-1}{2}M^2\right)^{\frac{k}{k-1}} \tag{78}$$

$$\frac{T_0}{T} = 1 + \frac{k-1}{2}M^2 \tag{79}$$

$$\frac{\rho_0}{\rho} = \left(1 + \frac{k-1}{2}M^2\right)^{\frac{1}{k-1}} \tag{80}$$

$$\frac{A}{A^*} = \frac{1}{M}\left(\frac{1 + \frac{k-1}{2}M^2}{\frac{k+1}{2}}\right)^{\frac{k+1}{2(k-1)}}. \tag{81}$$

Equation (78) provide property relations in terms of the local Mach number, the stagnation conditions, and critical conditions.