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Exercises

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Contents

Lecture 1: Fourier Theory

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Problem 1.1

Consider the following periodic functions. Find possible values for the period, using the definition of periodic functions.

a) $f(x) = e^{\cos x}$
b) $f(x) = \cos\left(x + \frac{\pi}{3}\right)$
c) $f(x) = \cos\left(\frac{\pi x}{3}\right)$
d) $f(x) = \cos x + \cos(3x)$
e) $f(x) = \cos(x)\cos(3x)$
f) $f(x) = \cos x + \cos(0{,}6x)$

Derivation

a) From the lecture (p. 1), we know that a function f is periodic, with period p, if

$$f(x) = f(x+p).$$

Using this definition we obtain:

$$e^{\cos x} = e^{\cos(x+p)}$$
$$\cos x = \cos\left(x+p\right)$$
$$\implies p = 2\pi, 4\pi, 6\pi, \ldots.$$

b) We use the same definition of periodicity to obtain

$$\cos\left(x + \frac{\pi}{3}\right) = \cos\left(x + p + \frac{\pi}{3}\right)$$
$$\implies p = 2\pi, 4\pi, 6\pi.$$

c) Following the same procedure we get

$$\cos\left(\frac{\pi x}{3}\right) = \cos\left(\frac{\pi\left(x+p\right))}{3}\right)$$
$$\cos\left(\frac{\pi x}{3}\right) = \cos\left(\frac{\pi x}{3} + \frac{\pi p}{3}\right)$$
$$\implies \frac{\pi p}{3} = 2\pi, 4\pi, 6\pi, \ldots$$
$$\implies p = 6, 12, 18, \ldots.$$

d) We follow the same procedure and get

$$\cos x + \cos(3x) = \cos\left(x+p\right) + \cos\left(3(x+p)\right)$$
$$\cos x + \cos(3x) = \cos(x+p) + \cos\left(3x + 3p\right).$$

This holds if p is an integer multiple of 2π and $3p$ is an integer multiple of 2π, i.e. $p = 2\pi, 4\pi, 6\pi, \ldots$

e) Once again, we follow the procedure and obtain

$$\cos(x)\cos(3x) = \cos(x+p)\cos(3(x+p))$$
$$\cos(x)\cos(3x) = \cos(x+p)\cos(3x+3p).$$

From the above, we see that $p = \pi, 2\pi, 3\pi$. This is due to the fact, that the two functions are multiplied, hence $\cos(0)\cdot\cos(0) = \cos(\pi)\cos(3\pi) = 1$

f) We follow the same procedure as in the previous subproblems and obtain

$$\cos x + \cos(0{,}6x) = \cos(x+p) + \cos\left(0{,}6x + 0{,}6p\right).$$

I.e. p must be an integer multiple of 2π and $0{,}6p$ must be an integer multiple of 2π. Hence $p = 10\pi, 20\pi, 30\pi, \dots$

a)	$p = 2\pi, 4\pi, 6\pi, \dots$
b)	$p = 2\pi, 4\pi, 6\pi, \dots$
c)	$p = 6, 12, 18, \dots$
d)	$p = 2\pi, 4\pi, 6\pi, \dots$
e)	$p = \pi, 2\pi, 3\pi, \dots$
f)	$p = 10\pi, 20\pi, 30\pi, \dots$

RESULT

Problem 1.2

Let

$$f(x) = x, \quad -1 \le x \le 1, \quad f(x) = f(x+p), \quad p = 2.$$

a) Sketch f

b) Find a Fourier series representation of f.

Derivation

a) On Figure 0.1 the periodic repeating function $f(x)$ is depicted.

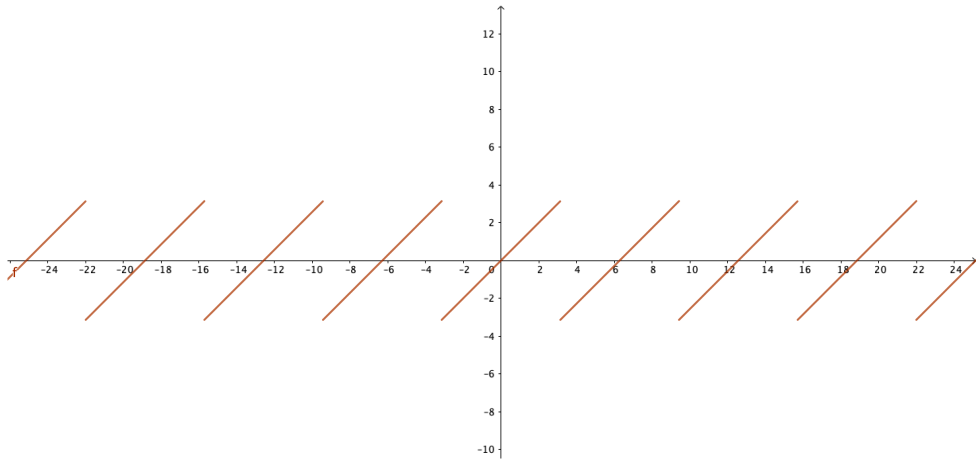


Figure 0.1: Sketch of f.

b) From the lecture (p. 5), we have the general representation of a Fourier series as

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right)$$

for

$$\begin{aligned} a_0 &= \frac{1}{2L} \int_{-L}^{L} f(x)\, \mathrm{d}x \\ a_n &= \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) \mathrm{d}x, n = 1, 2, 3, \dots \\ b_n &= \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x, n = 1, 2, 3, \dots. \end{aligned}$$

In this case, $L = \frac{p}{2} \implies L = 1$. Therefore:

$$\begin{aligned} a_0 &= \frac{1}{2} \int_{-1}^{1} x\, \mathrm{d}x \\ &= \frac{1}{2} \left[\frac{x^2}{2} \right]_{-1}^{1} \\ &= \frac{1}{2} \left(\frac{1}{2} - \frac{1}{2} \right) \\ &= 0 \\ a_n &= \int_{-1}^{1} x \cos(n\pi x)\, \mathrm{d}x \end{aligned}$$

$$
\begin{aligned}
&= 0 \\
b_n &= \int_{-1}^{1} x \sin(n\pi x)\, \mathrm{d}x \\
&= \left[\frac{\sin(\pi n x)}{\pi^2 n^2} - \frac{x\cos(\pi n x)}{\pi n} \right]_{-1}^{1} \\
&= \left[-\frac{x\cos(n\pi x)}{\pi n} \right]_{-1}^{1} \\
&= \left(-\frac{\cos(n\pi)}{\pi n} + \left(\frac{\cos(-n\pi)}{\pi n} \right) \right) \\
&= \left(-\frac{\cos(n\pi)}{\pi n} - \frac{\cos(n\pi)}{\pi n} \right) \\
&= -\frac{2}{\pi n} \cos(\pi n) .
\end{aligned}
$$

Hence, the complete Fourier representation is

$$
f(x) = \sum_{n=1}^{\infty} -\frac{2}{\pi n} \cos(\pi n) \sin(n\pi x) .
$$

a) See Figure 0.1

b) $f(x) = \sum_{n=1}^{\infty} -\frac{2}{\pi n} \cos(\pi n) \sin(n\pi x) .$

RESULT

Problem 1.3

Let $a \neq 0$ and $b \neq 0$. Let $f(x) = f(x + p_0)$ be a periodic function with period p_0. Is $f(ax + b)$ periodic? If yes, find a period. Give arguments for your answer.

Derivation

We let $f(x) = f(x + p)$ denote the function given in the assignment and $g(x) = f(ax + b)$ be the function which we wish to determine is periodic or not. To determine of $g(x)$ is periodic we must determine if $g(x) = g(x + p)$ holds. We get:

$$\begin{aligned} g(x+p) = g(x) \\ &= f(a(x+p)+b) \\ &= f(ax+b+ap). \end{aligned}$$

Comparing this with the original function f we see that it repeats when $ap = p_0$. Since $a \neq 0$ we can divide by a and obtain

$$p = \frac{p_0}{a}.$$

The function is periodic with period $p = \dfrac{p_0}{a}$ RESULT

Problem 1.4

Calculate the left-hand limit and the right hand limit of

$$f(x) = \frac{|x|}{x}$$

at $x = 0$.

Derivation

From the lecture (p. 2), we know that the left hand limit of $f(x)$ at x_0 is defined as

$$f(x_0-) = \lim_{h\to 0} f(x_0 - h)$$

and the right hand limit of $f(x)$ at x_0 is

$$f(x_0+) = \lim_{h\to 0} f(x_0 + h)$$

for positive h.

Using these two definitions we get:

$$\begin{aligned} f(0-) &= \lim_{h\to 0} f(0-h) \\ &= \lim_{h\to 0} \frac{|-h|}{-h} \\ &= \lim_{h\to 0} \frac{|h|}{-h} \\ &= -1 \\ f(0+) &= \lim_{h\to 0} f(0+h) \\ &= \lim_{h\to 0} \frac{|h|}{h} \\ &= 1. \end{aligned}$$

$$\boxed{\begin{aligned} f(0-) &= -1 \\ f(0+) &= 1. \end{aligned}}$$

Result

Problem 1.5

Calculate the left-hand derivative and the right-hand derivative of

$$f(x) = x|x|$$

at $x = 0$.

Derivation

From the lecture (p. 4) the definition of the left hand derivative of $f(x)$ at x_0 is defined as

$$f'(x_0-) = \lim_{h\to 0} \frac{f(x_0) - f(x_0 - h)}{h}$$

and the right hand derivative of $f(x)$ at x_0 is defined as

$$f'(x_0+) = \lim_{h\to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

for positive h.

Using these definitions we get:

$$\begin{aligned}
f'(0-) &= \lim_{h\to 0} \frac{f(0) - f(-h)}{h} \\
&= \lim_{h\to 0} \frac{-(-h|-h|)}{h} \\
&= \lim_{h\to 0} \frac{h|h|}{h} \\
&= \lim_{h\to 0} |h| \\
&= 0 \\
f'(0+) &= \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} \\
&= \lim_{h\to 0} \frac{h|h|}{h} \\
&= \lim_{h\to 0} |h| \\
&= 0.
\end{aligned}$$

$$\boxed{\begin{aligned} f'(0-) &= 0 \\ f'(0+) &= 0. \end{aligned}}$$

RESULT

Lecture 2: Even and odd functions

5. September 2025

Problem 2.1

Find a period and the corresponding Fourier coefficients of

a) $f(x) = \sin\left(\frac{\pi}{4}\right)\sin\left(\frac{2x}{4}\right) + \cos\left(\frac{\pi}{5}\right)\sin\left(\frac{3x}{7}\right)$
b) $f(x) = \sin\left(\sqrt{2}x\right) + \cos\left(\frac{x}{\sqrt{2}}\right)$
c) $f(x) = \sin\left(\frac{\pi}{3} + \frac{3\pi x}{4}\right)$.

Hint: You may want to use

$$\sin\left(\alpha+\beta\right) = \sin\alpha\cos\beta + \sin\beta\cos\alpha.$$

Derivation

a) We start by rewriting the given function as

$$f(x) = \sin\left(\frac{\pi}{4}\right)\sin\left(\frac{7\pi x}{14\pi}\right) + \cos\left(\frac{\pi}{5}\right)\sin\left(\frac{6\pi x}{14\pi}\right).$$

Now, comparing with the general form of a Fourier series:

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{n\pi x}{L}\right) + b_n\sin\left(\frac{n\pi x}{L}\right)\right)$$

We see that $L = 14\pi$, $p = 28\pi$, $b_6 = \cos(\pi/5)$ and $b_7 = \sin(\pi/4)$. All other Fourier coefficients are 0.

b) We rewrite as:

$$f(x) = \sin\left(\frac{2\pi x}{\sqrt{2}\pi}\right) + \cos\left(\frac{\pi x}{\sqrt{2}\pi}\right).$$

By comparison with the general form we see that $L = \sqrt{2}\pi$, $p = 2\sqrt{2}\pi$, $b_2 = 1$, and $a_1 = 1$. All other Fourier coefficients are 0.

c) We use the hint to rewrite as

$$\begin{aligned} f(x) &= \sin\left(\frac{\pi}{3} + \frac{3\pi x}{4}\right) \\ &= \sin\left(\frac{\pi}{3}\right)\cos\left(\frac{3\pi x}{4}\right) + \cos\left(\frac{\pi}{3}\right)\sin\left(\frac{3\pi x}{4}\right). \end{aligned}$$

By comparison, we see that $L = 4$, $p = 8$, $a_3 = \sin(\pi/3)$, and $b_3 = \cos(\pi/3)$. All other Fourier coefficients are zero

a)	$p = 28\pi$,	$b_6 = \cos\left(\frac{\pi}{5}\right)$,	$b_7 = \sin\left(\frac{\pi}{4}\right)$,
b)	$p = 2\sqrt{2}\pi$,	$a_1 = 1$,	$b_2 = 1$,
c)	$p = 8$,	$a_3 = \sin\left(\frac{\pi}{3}\right)$,	$b_3 = \cos\left(\frac{\pi}{3}\right)$.

RESULT

Problem 2.2

Are the following functions even or odd? Give arguments.

a) $f(x) = \left(\cos\left(x^2\right) + \cos(x)\right)\sin(x)$
b) $f(x) = \frac{1+\cos x}{2+\cos(x^2)}$
c) $f(x) = \frac{1+\sin(x)}{2+\sin(x^2)}$
d) f(x) = x |x|.

Derivation

From the lecture (p. 3), we know that a function f is even if

$$f(x) = f(-x) \quad \text{for all } x$$

and that a function f is odd if

$$f(x) = -f(-x) \quad \text{for all } x.$$

We will therefore try to insert $-x$ in the place of x and see if that causes a change of sign for each of the given functions.

We will however, instead utilize the simple fact that multiplication of functions follow the same rules for even/odd as multiplication of numbers. Furthermore, we know that x is odd, cos is even, sin is odd and $|x|$ is even. In the following, odd functions are called O and even functions are called E.

a) Using the method described above we get:

$$\begin{aligned} f(x) &= (\cos(x \cdot x) + \cos(x))\sin(x) \\ &= \cos(x \cdot x)\sin x + \cos(x)\sin(x) \\ &= E(O \cdot O) \cdot O + E \cdot O \\ &= E \cdot O + E \cdot O \\ &= O + O \\ &= O. \end{aligned}$$

b) Again,

$$\begin{aligned} f(x) &= \frac{1+\cos x}{2+\cos(x \cdot x)} \\ &= \frac{E+E}{E+E} \\ &= \frac{E}{E} \\ &= E. \end{aligned}$$

c) Again,

$$\begin{aligned} f(x) &= \frac{1+\sin x}{2+\sin(x \cdot x)} \\ &= \frac{E+O}{E+O(E)} \\ &= \frac{E+O}{E+E} \end{aligned}$$

$$
\begin{aligned}
&= \frac{E+O}{E} \\
&= \frac{E}{E} + \frac{O}{E} \\
&= E + O.
\end{aligned}
$$

As the sum of an even and an odd function is neither even nor odd if none of the functions are 0. Hence the function in question is neither even nor odd.

d) Lastly,

$$
\begin{aligned}
f(x) &= x \cdot |x| \\
&= O \cdot E \\
&= O.
\end{aligned}
$$

RESULT

a) Odd,
b) Even,
c) Neither even nor odd,
d) Odd.

Problem 2.3

Consider the periodic function

$$f(x) = \begin{cases} x, & -2 < x < -1 \\ x + k, & -1 < x < 1 \\ x, & 1 < x < 2 \end{cases}$$
$$f(x) = f(x+p), p = 4$$

where k is a constant. Decompose $f(x)$ into its even and odd part.

Derivation

For $-2 < x < -1$ we get

$$f_1(x) = \frac{1}{2}\big(f(x) + f(-x)\big) = \frac{1}{2}(x + (-x)) = 0$$
$$f_2(x) = \frac{1}{2}\big(f(x) - f(-x)\big) = \frac{1}{2}(x - (-x)) = x.$$

For $-1 < x < 1$ we get

$$f_1(x) = \frac{1}{2}\big(f(x) + f(-x)\big) = \frac{1}{2}(x + k + (-x + k)) = k$$
$$f_2(x) = \frac{1}{2}\big(f(x) - f(-x)\big) = \frac{1}{2}(x + k - (-x + k)) = x.$$

And for $1 < x < 2$ we get

$$f_1(x) = \frac{1}{2}\big(f(x) + f(-x)\big) = \frac{1}{2}(x + (-x)) = 0$$
$$f_2(x) = \frac{1}{2}\big(f(x) - f(-x)\big) = \frac{1}{2}(x - (-x)) = x.$$

We thus have the decomposed function $f(x)$

$$f(x) = f_1(x) + f_2(x)$$

with

$$f_1(x) = \begin{cases} 0, & -2 < x < -1 \\ k, & -1 < x < 1 \\ 0, & 1 < x < 2. \end{cases}$$
$$f_1(x) = f_1(x+p), \quad p = 4$$

and

$$f_2(x) = x, \quad -2 < x < 2$$
$$f_2(x) = f_2(x+p), \quad p = 4.$$

$$\boxed{\begin{aligned} &f(x) = f_1(x) + f_2(x) \\ &f_1(x) = \begin{cases} 0, & -2 < x < -1 \\ k, & -1 < x < 1 \\ 0, & 1 < x < 2. \end{cases} \\ &f_1(x) = f_1(x+p), \quad p = 4 \\ &f_2(x) = x, \quad -2 < x < 2 \\ &f_2(x) = f_2(x+p), \quad p = 4. \end{aligned}}$$

RESULT

Problem 2.4

Consider the periodic function

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ 2x, & 0 < x < \pi \end{cases}$$

$$f(x) = f(x+p), p = 2\pi.$$

Decompose $f(x)$ into its even and odd part and find its Fourier series.

Hint: You may want to use examples from the lecture.

Derivation

For $-\pi < x < 0$ we get

$$f_1(x) = \frac{1}{2}\big(f(x) + f(-x)\big) = \frac{1}{2}(0 + (-2x)) = -x$$
$$f_2(x) = \frac{1}{2}\big(f(x) - f(-x)\big) = \frac{1}{2}(0 - (-2x)) = x.$$

For $0 < x < \pi$ we get

$$f_1(x) = \frac{1}{2}\big(f(x) + f(-x)\big) = \frac{1}{2}(2x + (0)) = x$$
$$f_2(x) = \frac{1}{2}\big(f(x) - f(-x)\big) = \frac{1}{2}(2x - (0)) = x.$$

Therefore

$$f_1(x) = |x|, -\pi < x < \pi.$$

From the lecture (p. 7) the Fourier cosine series of this is

$$f_1(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi n^2}(\cos(n\pi) - 1)\cos(nx).$$

Also

$$f_2(x) = x, -\pi < x < \pi.$$

Which has the Fourier sine series (p. 6)

$$f_2(x) = \sum_{n=1}^{\infty} (-1)^{n+1}\frac{2}{n}\sin(nx).$$

Combining these two Fourier series we get the complete Fourier series:

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty}\left(\frac{2}{\pi n^2}(\cos(n\pi - 1))\cos(nx) + (-1)^{n+1}\frac{2}{n}\sin(nx)\right).$$

$$f_1(x) = |x|, \quad -\pi < x < \pi$$
$$f_2(x) = x, -\pi < x < \pi$$
$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty}\left(\frac{2}{\pi n^2}(\cos(n\pi - 1))\cos(nx) + (-1)^{n+1}\frac{2}{n}\sin(nx)\right).$$

Result

Problem 2.5

Decompose

$$f(x) = e^{ix}$$

into its even and odd part.

Hint; you may want to use the representation of $f(x)$ in terms of $\cos x$ and $\sin x$.

Derivation

We rewrite the function as

$$f(x) = e^{ix} = \cos x + i \sin x.$$

Now, to decompose we follow the same procedure as above:

$$\begin{aligned}
f_1(x) &= \frac{1}{2}\big(f(x) + f(-x)\big) \\
&= \frac{1}{2}(\cos x + i \sin x + \cos(-x) + i \sin(-x) \\
&= \frac{1}{2}(\cos x + i \sin x + \cos x - i \sin(x)) \\
&= \cos x \\
f_2(x) &= \frac{1}{2}\big(f(x) - f(-x)\big) \\
&= \frac{1}{2}(\cos x + i \sin x - \cos(-x) - i \sin(-x)) \\
&= \frac{1}{2}(\cos x + i \sin x - \cos x + i \sin x) \\
&= i \sin x.
\end{aligned}$$

$$\boxed{\begin{aligned} f_1(x) &= \cos x \\ f_2(x) &= i \sin x. \end{aligned}}$$

RESULT

Lecture 3: Fourier Series

12. September 2025

Problem 3.1

Consider the function $f(x) = \cos x, 0 < x < \pi$.

a) Sketch the expansion of f and the corresponding Fourier series.
b) Sketch the even expansion of f and find the corresponding Fourier cosine series.
c) Sketch the odd expansion of f and find the corresponding Fourier cosine series.
d) Calculate the error E_N of the expansion and the odd expansion for $N = 1, 2, 3$. What can be said about the quality of the approximations for these two expansions=

Hint: In some cases the expansion can be an even/odd function.

Hint: For $m, n \in \mathbb{N}, m \neq n, L > 0$:

$$\int_0^{\frac{\pi}{2}} \cos x \sin(2nx)\, \mathrm{d}x = -\frac{2n}{1-4n^2}$$

$$\int_0^L \cos\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x = \frac{nL}{\pi\left(m^2 - n^2\right)} \left(\cos(n\pi)\cos(m\pi) - 1\right)$$

$$\int_0^L \cos\left(\frac{\pi x}{L}\right) \sin\left(\frac{\pi x}{L}\right) \mathrm{d}x = 0.$$

Derivation

a) The expansion of f is depicted on Figure 0.2. Based on the figure the expansion looks to generate an odd

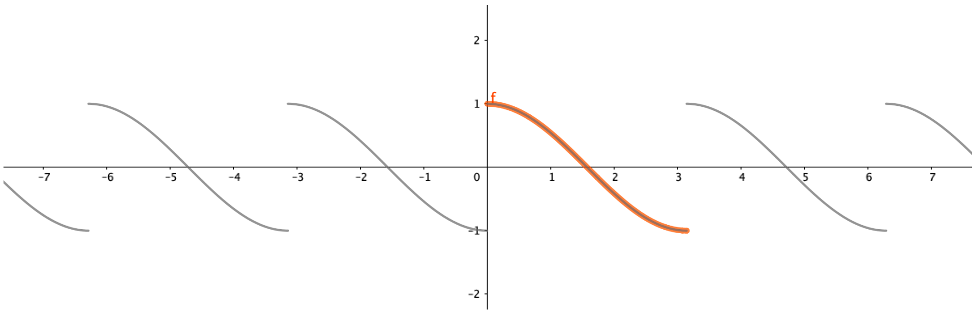


Figure 0.2: Expansion of f.

function. We check that this is the case using the definition of odd functions (Lec. 2, p. 3):

$$f(x) = -f(-x), \text{ for all } x.$$

For $-\pi/2 < x < 0$ inputting $-x$ in the place of x gives:

$$f(x) = -\cos x$$
$$f(-x) = -\cos(-x)$$
$$f(-x) = \cos x$$
$$f(-x) = -f(x).$$

And for $0 < x < \pi/2$ we get

$$\begin{aligned} f(x) &= \cos x \\ f(-x) &= \cos(-x) \\ f(-x) &= -\cos x \\ f(-x) &= -f(x). \end{aligned}$$

Hence, the expansion of f is odd and the corresponding Fourier series will therefore be a Fourier sine series. The period is $p = \pi \implies L = \pi/2$. From Lecture 2 (p. 5) the Fourier sine series is defined as

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

for

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x.$$

We start by determining the Fourier coefficients b_n as

$$\begin{aligned} b_n &= \frac{4}{\pi}\int_0^{\frac{\pi}{2}} \cos x \sin\left(\frac{2n\pi x}{\pi}\right)\mathrm{d}x \\ &= \frac{4}{\pi}\int_0^{\frac{\pi}{2}} \cos(x)\sin(2nx)\,\mathrm{d}x \\ &= \frac{8n}{\pi\left(4n^2-1\right)}. \end{aligned}$$

Hence,

$$f(x) = \sum_{n=1}^{\infty} \frac{8n}{\pi\left(4n^2-1\right)}\sin(2nx).$$

b) The even expansion of f is depicted on Figure 0.3. The even expansion of $\cos x, 0 < x < \pi$ is simply $\cos x$

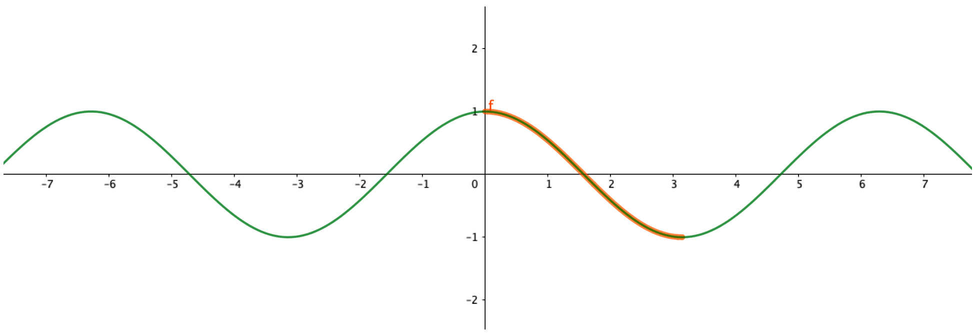


Figure 0.3: Even expansion of f.

and the Fourier series can therefore be found by comparison:

$$\begin{aligned} f(x) &= \cos x \\ f(x) &= a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) \\ \implies \quad & a_0 = 0 \quad a_1 = 1. \end{aligned}$$

c) The odd expansion of f is depicted on Figure 0.4. Per definition, the odd expansion results in an odd function. Therefore, we must find a Fourier sine series. From Lecture 2 (p. 5) this is defined as

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

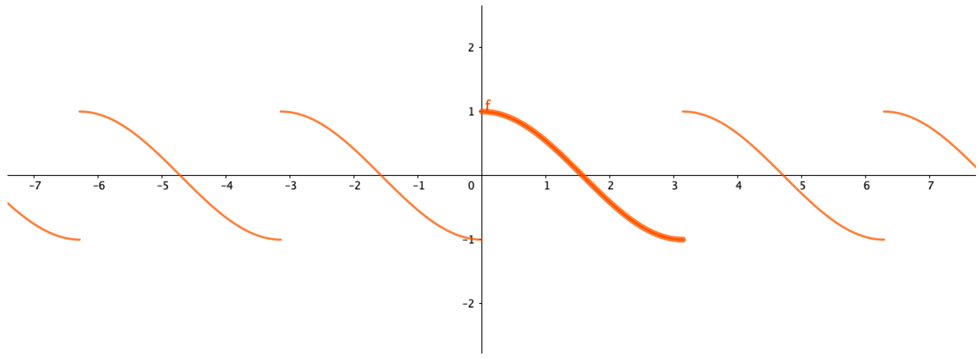


Figure 0.4: Odd expansion of f.

for

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x.$$

We start by determining the Fourier coefficients b_n as:

$$\begin{aligned}
b_n &= \frac{2}{\pi}\int_0^\pi f(x)\sin\left(\frac{n\pi x}{\pi}\right) \\
&= \frac{2}{\pi}\int_0^\pi \cos x\sin(nx)\,\mathrm{d}x \\
&= \frac{2n}{\pi(1-n^2)}\left(\cos(n\pi)\cos(\pi)-1\right).
\end{aligned}$$

From the hints we know that this is 0 for $n=1$, hence:

$$f(x) = \sum_{n=1}^\infty b_n\sin\left(\frac{n\pi x}{L}\right) = \sum_{n=2}^\infty \frac{2n}{\pi\left(n^2-1\right)}(1-\cos(n\pi))\sin(nx).$$

d) From the lecture (p. 5), the error E_N is defined as

$$E_N = \int_{-L}^L f^2(x)\,\mathrm{d}x - L\left(2a_0^2 + \sum_{n=1}^N\left(a_n^2+b_n^2\right)\right).$$

For the expansion we get

$$\begin{aligned}
E_N &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2(x)\,\mathrm{d}x - \frac{\pi}{2}\left(\sum_{n=1}^N\left(\frac{8n}{\pi\left(4n^2-1\right)}\right)^2\right) \\
&= \frac{\pi}{2} - \frac{\pi}{2}\left(\sum_{n=1}^N\left(\frac{64n^2}{\pi^2\left(4n^2-1\right)^2}\right)\right) \\
&= \frac{\pi}{2} - \frac{32}{\pi}\sum_{n=1}^N\frac{n^2}{\left(4n^2-1\right)^2} \\
E_1 &= \frac{\pi}{2} - \frac{32}{\pi}\cdot\frac{1}{9} = 0{,}439 \\
E_2 &= \frac{\pi}{2} - \frac{32}{\pi}\cdot\left(\frac{1}{9}+\frac{4}{15^2}\right) = 0{,}257\,944\,885\,1 \\
E_3 &= \frac{\pi}{2} - \frac{32}{\pi}\cdot\left(\frac{1}{9}+\frac{4}{15^2}+\frac{9}{35^2}\right) = 0{,}183\,109\,581\,3.
\end{aligned}$$

And for the odd expansion we get:

$$\begin{aligned}
E_N^{(o)} &= \int_{-\pi}^{\pi} \cos^2(x)\,\mathrm{d}x - \pi\left(\sum_{n=2}^N\left(\frac{2n}{\pi\left(n^2-1\right)}(1+\cos(n\pi))\right)^2\right) \\
&= \pi - \pi\left(\sum_{n=2}^N\frac{4n^2}{\pi^2\left(n^2-1\right)^2}(1+\cos(\pi n))^2\right)
\end{aligned}$$

$$
\begin{aligned}
&= \pi - \frac{4}{\pi}\left(\sum_{n=2}^{N} \frac{n^2}{\left(n^2-1\right)^2}\left(1+\cos(\pi n)\right)^2\right) \\
E_1^{(o)} &= \pi - \frac{4}{\pi}\cdot 0 = \pi \\
E_2^{(o)} &= \pi - \frac{4}{\pi}\cdot\left(0 + \frac{4}{9}\cdot 4\right) = 0{,}878\,055\,685\,2 \\
E_3^{(o)} &= \pi - \frac{4}{\pi}\cdot\left(0 + \frac{16}{9} + \frac{9}{64}\cdot 0\right) = 0{,}878\,055\,685\,2.
\end{aligned}
$$

All in all, the expansion seems to work better than the odd expansion even when taking into account the difference in interval lengths.

RESULT

a) $f(x) = \sum_{n=1}^{\infty} \frac{8n}{\pi\left(4n^2-1\right)}\sin(2nx)$

b) $f(x) = \cos x$

c) $f(x) = \sum_{n=2}^{\infty} \frac{2n}{\pi\left(n^2-1\right)}\left(1-\cos(n\pi)\right)\sin(nx).$

Problem 3.2

Find the Fourier integral representation of

$$f(x) = \begin{cases} 1, & -1 < x < 1 \\ -1, & 1 < x < 2 \\ -1, & -2 < x < -1 \\ 0, & \text{otherwise} \end{cases}.$$

Derivation

From the lecture (p. 8) the Fourier integral is represented as

$$f(x) = \int_0^{\infty} (A(\omega)\cos(\omega x) + B(\omega)\sin(\omega x))\, \mathrm{d}\omega$$

with

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos(\omega x)\, \mathrm{d}\omega$$
$$B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin(\omega x)\, \mathrm{d}\omega.$$

We determine the Fourier coefficients as:

$$\begin{aligned} A(\omega) &= \frac{1}{\pi}\left(\int_{-2}^{-1} -1\cdot\cos(\omega x)\, \mathrm{d}\omega + \int_{-1}^{1} 1\cdot\cos(\omega x)\, \mathrm{d}\omega + \int_{1}^{2} -1\cdot\cos(\omega x)\, \mathrm{d}\omega\right) \\ &= \frac{1}{\pi}\left(\int_{-1}^{1}\cos(\omega x)\, \mathrm{d}\omega - \int_{-2}^{-1}\cos(\omega x)\, \mathrm{d}\omega - \int_{1}^{2}\cos(\omega x)\, \mathrm{d}\omega\right) \\ &= \frac{4\sin\omega - 2\sin(2\omega)}{\omega} \\ B(\omega) &= \frac{1}{\pi}\left(\int_{-1}^{1}\sin(\omega x)\, \mathrm{d}\omega - \int_{-2}^{-1}\sin(\omega x)\, \mathrm{d}\omega - \int_{1}^{2}\sin(\omega x)\, \mathrm{d}\omega\right) \\ &= 0. \end{aligned}$$

Hence, the Fourier integral representation is:

$$\boxed{f(x) = \int_0^{\infty} \frac{4\sin\omega - 2\sin(2\omega)}{\omega}\cos(\omega x)\, \mathrm{d}\omega}$$ RESULT

Problem 3.3

Find the Fourier integral representation of

$$f(x) = \begin{cases} -1, & -2 < x < 0 \\ 1, & 0 < x < 2 \\ 0, & \text{otherwise} \end{cases}.$$

Derivation

Here, we follow the same procedure as above:

$$\begin{aligned} A(\omega) &= \frac{1}{\pi}\left(\int_0^2 \cos(\omega x)\,\mathrm{d}\omega - \int_{-2}^0 \cos(\omega x)\,\mathrm{d}\omega\right) \\ &= 0 \\ B(\omega) &= \frac{1}{\pi}\left(\int_0^2 \sin(\omega x)\,\mathrm{d}\omega - \int_{-2}^0 \sin(\omega x)\,\mathrm{d}\omega\right) \\ &= \frac{1}{\pi}\cdot\frac{4\sin^2(\omega)}{\omega} \\ &= \frac{4\sin^2(\omega)}{\pi\omega}. \end{aligned}$$

Hence, the Fourier integral representation is:

$$\boxed{f(x) = \int_0^\infty \frac{4\sin^2(\omega)}{\omega\pi}\sin(\omega x)\,\mathrm{d}\omega} \qquad \text{RESULT}$$

Problem 3.4

Consider the periodic rectangular wave and its Fourier series as discussed in Lecture 1. Calculate the error of the N-th approximation for $N = 1, 2, 3$.

Derivation

From Lecture 1 pp. 5-6 the periodic rectangular wave is given by

$$f(x) = \begin{cases} -k, & -\pi < x < 0 \\ k, & 0 < x < \pi. \end{cases}$$
$$f(x) = f(x + 2\pi), \quad p = 2L = 2\pi.$$

And its Fourier coefficients are

$$a_0 = 0$$
$$a_n = 0$$
$$b_n = \frac{2k}{n\pi}(1 - \cos(n\pi)).$$

From Problem 3.1 the error E_N is defined as

$$E_N = \int_{-L}^{L} f^2(x)\,\mathrm{d}x - L\left(2a_0^2 + \sum_{n=1}^{N}\left(a_n^2 + b_n^2\right)\right).$$

Which corresponds to:

$$\begin{aligned} E_N &= \int_{-\pi}^{\pi} k^2\,\mathrm{d}x - \pi\left(\sum_{n=1}^{N}\left(\frac{2k}{n\pi}(1 - \cos(n\pi))\right)^2\right) \\ &= 2\pi k^2 - \pi\left(\sum_{n=1}^{N}\frac{4k^2}{n^2\pi^2}(1 - \cos(n\pi))^2\right) \\ &= 2\pi k^2 - \frac{4k^2}{\pi}\left(\sum_{n=1}^{N}\frac{1}{n^2}(1 - \cos(n\pi))^2\right) \\ E_1 &= 2\pi k^2 - \frac{4k^2}{\pi}\cdot 4 = 1{,}19k^2 \\ E_2 &= E_1 \\ E_3 &= 2\pi k^2 - \frac{4k^2}{\pi}\cdot\left(4 + \frac{4}{9}\right) = 0{,}624\,342\,886\,1k^2. \end{aligned}$$

$$\boxed{\begin{aligned} E_1 &= 1{,}19k^2 \\ E_2 &= E_1 \\ E_3 &= 0{,}642k^2. \end{aligned}}$$

RESULT

Lecture 4: Fourier sine and cosine transformations

19. September 2025

Problem 4.1

Represent $f(x)$, $0 < x < \infty$, defined as

$$f(x) = \begin{cases} \cos x, & 0 < x < \pi \\ 0, & x > \pi. \end{cases}$$

as a Fourier sine integral. *Hint:*

$$\int_0^{\pi} \cos x \sin(\omega x)\, \mathrm{d}x = \frac{\omega(1+\cos(\omega\pi))}{\omega^2 - 1}$$

Derivation

From the Lecture p. 1 the Fourier sine integral is defined as

$$f(x) = \int_0^{\infty} B(\omega) \sin(\omega x)\, \mathrm{d}\omega$$

with

$$B(\omega) = \frac{2}{\pi} \int_0^{\infty} f(x) \sin(\omega x)\, \mathrm{d}x.$$

Plugging in the $f(x)$ from the problem we get

$$\begin{aligned} B(\omega) &= \frac{2}{\pi} \int_0^{\pi} \cos x \sin \omega(x)\, \mathrm{d}x \\ &= \frac{2\omega(1+\cos(\omega\pi))}{\pi\left(\omega^2 - 1\right)}. \end{aligned}$$

Hence,

$$\boxed{f(x) = \int_0^{\infty} \frac{2\omega(1+\cos(\omega\pi))}{\pi\left(\omega^2 - 1\right)} \sin(\omega x)\, \mathrm{d}\omega} \qquad \text{RESULT}$$

Problem 4.2

Represent $f(x)$, $0 < x < \infty$, defined as

$$f(x) = \begin{cases} \sin x, & 0 < x < \pi \\ 0, & x > \pi \end{cases}$$

as a Fourier cosine integral. *Hint:*

$$\int_0^{\pi} \sin x \cos(\omega x)\,\mathrm{d}x = \frac{1+\cos(\omega\pi)}{1-\omega^2}.$$

Derivation

The Fourier cosine integral is in the Lecture p. 1 given as

$$f(x) = \int_0^{\infty} A(\omega)\cos(\omega x)\,\mathrm{d}\omega$$

with

$$A(\omega) = \frac{2}{\pi}\int_0^{\infty} f(x)\cos(\omega x)\,\mathrm{d}x.$$

As was the case for problem 4.1 we find $A(\omega)$ as

$$\begin{aligned} A(\omega) &= \frac{2}{\pi}\int_0^{\pi} \sin x \cos(\omega x)\,\mathrm{d}x \\ &= \frac{2(1+\cos(\omega\pi))}{\pi\left(1-\omega^2\right)}. \end{aligned}$$

Thus,

$$\boxed{f(x) = \int_0^{\infty} \frac{2(1+\cos(\omega\pi))}{\pi\left(1-\omega^2\right)}\cos(\omega x)\mathrm{d}\omega} \qquad \text{RESULT}$$

Problem 4.3

Find the Fourier cosine transform of $f(x), 0 < x < \infty$, defined as

$$f(x) = \begin{cases} 1, & 0 < x < 1 \\ -1 & 1 < x < 2 \\ 0, & x > 2 \end{cases}.$$

Derivation

From the Lecture (p. 4) the Fourier cosine transform is given as

$$\hat{f}_C(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos(\omega x) \, \mathrm{d}x.$$

We find $A(\omega)$ as

$$\begin{aligned} A(\omega) &= \sqrt{\frac{2}{\pi}} \left(\int_0^1 \cos(\omega x) \, \mathrm{d}x - \int_1^2 \cos(\omega x) \, \mathrm{d}x \right) \\ &= \sqrt{\frac{2}{\pi}} \frac{2\sin(\omega) - \sin(2\omega)}{\omega}. \end{aligned}$$

Hence,

$$\boxed{f(x) = \sqrt{\frac{2}{\pi}} \frac{2\sin\omega - \sin(2\omega)}{\omega} \cos(\omega x)}$$ RESULT

Problem 4.4

Find the Fourier transform of

$$f(x) = \begin{cases} e^{2ix}, & -1 < x < 1 \\ 0, & \text{otherwise.} \end{cases}.$$

Derivation

From the Lecture (p. 7) the Fourier transform is defined as

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} \, \mathrm{d}x.$$

By insertion we get:

$$\begin{aligned}
\hat{f}(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{2ix} e^{-i\omega x} \\
&= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{i(2-\omega)x} \, \mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{i(2-\omega)x}}{i(2-\omega)x} \right]_{-1}^{1} \\
&= \frac{1}{\sqrt{2\pi}} \left(\frac{e^{i(2-\omega)} - e^{-i(2-\omega)}}{i(2-\omega)} \right) \\
&= \frac{1}{\sqrt{2\pi}} \left(\frac{\cos(2-\omega) + i\sin(2-\omega) - \cos(\omega-2) - i\sin(\omega-2)}{i(2-\omega)} \right) \\
&= \frac{1}{\sqrt{2\pi}} \frac{2i\sin(2-\omega)}{i(2-\omega)} \\
&= \frac{1}{\sqrt{2\pi}} \frac{2\sin(2-\omega)}{2-\omega}.
\end{aligned}$$

$$\boxed{\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \frac{2\sin(2-\omega)}{2-\omega}}$$

RESULT

Problem 4.5

Use the formulae of the Fourier transform, Fourier cosine transform and Fourier sine transform for first and second derivatives from the Lecture to derive the corresponding formulae for third derivatives, i.e. for

$$\mathscr{F}_C\left(f'''\right), \mathscr{F}_S\left(f'''\right), \mathscr{F}\left(f'''\right).$$

Derivation

From the Lecture (pp. 5 & 8) the first and second derivatives of each of the Fourier transforms are:

$$\mathscr{F}_C(f') = \omega\mathscr{F}_S(f) - \sqrt{\frac{2}{\pi}}f(0) \qquad \mathscr{F}_C(f'') = -\omega^2\mathscr{F}_C(f) - \sqrt{\frac{2}{\pi}}f'(0)$$

$$\mathscr{F}_S(f') = -\omega\mathscr{F}_C(f) \qquad \mathscr{F}_S(f'') = -\omega^2\mathscr{F}_S(f) + \sqrt{\frac{2}{\pi}}\omega f(0)$$

$$\mathscr{F}(f') = i\omega\mathscr{F}(f) \qquad \mathscr{F}(f'') = -\omega^2\mathscr{F}(f).$$

Starting with the Fourier cosine transform, we will simply apply the first derivative to the second derivative as:

$$\begin{aligned}
\mathscr{F}_C(f''') &= \mathscr{F}_C(f'')' \\
&= \omega\mathscr{F}_S(f'') - \sqrt{\frac{2}{\pi}}f''(0) \\
&= \omega\left(-\omega^2\mathscr{F}_S(f) + \sqrt{\frac{2}{\pi}}\omega f(0)\right) - \sqrt{\frac{2}{\pi}}f''(0) \\
&= -\omega^3\mathscr{F}_S(f) + \sqrt{\frac{2}{\pi}}\omega^2 f(0) - \sqrt{\frac{2}{\pi}}f''(0).
\end{aligned}$$

Now we follow the same procedure for the Fourier sine and Fourier transforms as

$$\begin{aligned}
\mathscr{F}_S(f''') &= \mathscr{F}_S(f'')' \\
&= -\omega\mathscr{F}_C(f'') \\
&= -\omega\left(-\omega^2\mathscr{F}_C(f) - \sqrt{\frac{2}{\pi}}f'(0)\right) \\
&= \omega^3\mathscr{F}_C(f) + \sqrt{\frac{2}{\pi}}\omega f'(0) \\
\mathscr{F}(f''') &= \mathscr{F}(f'')' \\
&= i\omega\mathscr{F}(f'') \\
&= i\omega\left(-\omega^2\mathscr{F}(f)\right) \\
&= -i\omega^3\mathscr{F}(f).
\end{aligned}$$

$$\boxed{\begin{aligned}
\mathscr{F}_C(f''') &= -\omega^3\mathscr{F}_S(f) + \sqrt{\frac{2}{\pi}}\omega^2 f(0) - \sqrt{\frac{2}{\pi}}f''(0) \\
\mathscr{F}_S(f''') &= \omega^3\mathscr{F}_C(f) + \sqrt{\frac{2}{\pi}}\omega f'(0) \\
\mathscr{F}(f''') &= -i\omega^3\mathscr{F}(f).
\end{aligned}}$$

RESULT

Problem 4.6

Let

$$\hat{f}_S(\omega) = \begin{cases} 1, & 0 < \omega < 1 \\ 0, & \text{otherwise} \end{cases}.$$

Find $f(x)$.

Derivation

From the Lecture (p. 5) the inverse Fourier sine transform is given as:

$$\mathscr{F}_S^{-1}(\hat{f}_S(\omega)) = f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_S(\omega) \sin(\omega x) \, d\omega.$$

We substitute the $\hat{f}_S(\omega)$ given in the assignment into the above expression to get $f(x)$ as

$$\begin{aligned} f(x) &= \sqrt{\frac{2}{\pi}} \int_0^1 \sin(\omega x) \, d\omega \\ &= \sqrt{\frac{2}{\pi}} \frac{1 - \cos(x)}{x}. \end{aligned}$$

$$\boxed{f(x) = \sqrt{\frac{2}{\pi}} \frac{1 - \cos x}{x}}$$

RESULT

Problem 4.7

Let

$$\hat{f}(\omega) = \begin{cases} 1, & -1 < \omega < 1 \\ 0, & \text{otherwise} \end{cases}.$$

Find $f(x)$.

Derivation

From the Lecture (p. 7) the inverse Fourier transform is given as:

$$\mathscr{F}^{-1}(\hat{f}) = f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x} \, \mathrm{d}\omega.$$

Following the same logic as in the above problem we get

$$\begin{aligned} f(x) &= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{i\omega x} \, \mathrm{d}\omega \\ &= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{i\omega x}}{ix} \right]_{\omega=-1}^{\omega=1} \\ &= \frac{1}{\sqrt{2\pi}} \frac{e^{ix} - e^{-ix}}{ix} \\ &= \frac{1}{\sqrt{2\pi}} \frac{\cos(x) + i\sin(x) - \cos(-x) - i\sin(-x)}{ix} \\ &= \frac{1}{\sqrt{2\pi}} \frac{2\sin(x)}{x}. \end{aligned}$$

$$\boxed{f(x) = \frac{1}{\sqrt{2\pi}} \frac{2\sin x}{x}}$$

RESULT

Problem 4.8

Consider the ODE

$$f''(x) + f(x) = r(x)$$

where

$$r(x) = \begin{cases} 1, & 0 < x < 1 \\ -1, & 1 < x < 2 \\ 0, & x > 2 \end{cases}$$

and assume that we have the boundary condition $f'(0) = 0$. Apply the Fourier cosine transform to both sides of the ODE to derive the corresponding equation for $\hat{f}_C(\omega)$. Find $\hat{f}_C(\omega)$.

Hint: You may want to use the solution of Problem 4.3.

Derivation

We apply the Fourier cosine transform to both sides of the ODE as:

$$\mathscr{F}_C(f'' + f) = \mathscr{F}_C(r) \implies \mathscr{F}_C(f'') + \mathscr{F}_C(f) = \mathscr{F}_C(r).$$

Which corresponds to

$$\begin{aligned} -\omega^2 \hat{f}_C(\omega) + \hat{f}_C(\omega) &= \sqrt{\frac{2}{\pi}} \frac{2\sin\omega - \sin(2\omega)}{\omega} \\ \hat{f}_C(\omega) &= \sqrt{\frac{2}{\pi}} \frac{2\sin\omega - \sin(2\omega)}{\omega\left(1-\omega^2\right)}. \end{aligned}$$

$$\boxed{\hat{f}_C(\omega) = \sqrt{\frac{2}{\pi}} \frac{2\sin\omega - \sin(2\omega)}{\omega(1-\omega^2)}}$$ RESULT

Lecture 5: Partial Differential Equations

26. September 2025

Problem 5.1

Write down the general expression for a linear first order PDE for $u(x, y, t)$.

Derivation

From the Lecture (p. 1) the general expression for a linear first order PDE for $u(x, y)$ is

$$a_0(x,t) + a_1(x,t)u(x,t) + a_2(x,t)\frac{\partial u(x,t)}{\partial x} + a_3(x,t)\frac{\partial u(x,t)}{\partial t} = 0.$$

Extending this pattern we can write the general expression for a linear first order ODE for $u(x, y, t)$ as

$$\boxed{a_0(x,y,t) + a_1(x,y,t)u(x,y,t) + a_2(x,y,t)u_x(x,y,t) + a_3(x,y,t)u_y(x,y,t) + a_4(x,y,t)u_t(x,y,t) = 0}$$ RESULT

Problem 5.2

Write down the general expression for a linear second order PDE for $u(x, y)$. You may assume that $u(x, y)$ is twice continuously differentiable.

Derivation

Extending the logic from the previous answer, the general expression for a linear second order PDE for $u(x, y)$ is:

$$\boxed{\begin{aligned} &a_0(x,y) + a_1(x,y)u(x,y) + a_2(x,y)u_x(x,y) + a_3(x,y)u_y(x,y) \\ &\quad + a_4(x,y)u_{xx}(x,y) + a_5(x,y)u_{yy}(x,y) + a_6(x,y)u_{xy}(x,y) = 0 \end{aligned}} \qquad \text{RESULT}$$

Problem 5.3

Consider the general homogeneous first order PDE for $u(x,t)$ with solutions $u_1(x,t)$ and $u_2(x,t)$. Show that $u(x,t) = c_1 u_1(x,t) + c_2 u_2(x,t)$ is also a solution for arbitrary constants c_1 and c_2.

Derivation

From the Lecture (p. 2) the general homogeneous first order PDE for $u(x,t)$ is

$$a_1(x,t)u(x,t) + a_2(x,t)u_x(x,t) + a_3(x,t)u_t(x,t) = 0.$$

We will see if this holds for arbitrary constants c_1 and c_2 as

$$\begin{aligned}
&a_1(c_1u_1 + c_2u_2) + a_2\frac{\partial c_1u_1 + c_2u_2}{\partial x} + a_3\frac{\partial c_1u_1 + c_2u_2}{\partial t} \\
&= a_1c_1u_1 + a_1c_2u_2 + a_2\frac{\partial c_1u_1}{\partial x} + a_2\frac{\partial c_2u_2}{\partial x} + a_3\frac{\partial c_1u_1}{\partial t} + a_3\frac{\partial c_2u_2}{\partial t} \\
&= c_1\left(a_1u_1 + a_2\frac{\partial u_1}{\partial x} + a_3\frac{\partial u_1}{\partial t}\right) + c_2\left(a_1u_2 + a_2\frac{\partial u_2}{\partial x} + a_2\frac{\partial u_2}{\partial t}\right) \\
&= c_1(0) + c_2(0) \\
&= 0.
\end{aligned}$$

Hence $u(x,t) = c_1u_1(x,t) + c_2u_2(x,t)$ is also a solution of the general homogeneous first order PDE for $u(x,t)$.

Problem 5.4

Find the solution of the one-dimensional wave equation

$$u_{tt}(x,t) = u_{xx}(x,t), \quad 0 \le x \le 1$$

subject to the boundary conditions

$$u(0,t) = u(1,t) = 0, \quad \text{for all } t$$

and the initial condition

$$u(x,0) = k_0 \sin(3\pi x), \quad u_t(x,0) = 0, \quad 0 \le x \le 1,$$

where k_0 is a constant. You may start your calculations directly from the general solution given in the Lecture.

Derivation

From the Lecture (p. 8) the general solution to the one-dimensional wave equation is

$$u(x,t) = \sum_{n=1}^{\infty} \left(B_n \cos(\lambda_n t) + B_n^* \sin(\lambda_n t) \right) \sin(n\pi x) \quad \lambda = n\pi.$$

From the initial elongation (p. 9) we get

$$\begin{aligned} u(x,0) &= \sum_{n=1}^{\infty} B_n \sin(n\pi x) \\ &= k_0 \sin(3\pi x) \\ \implies \quad B_3 &= k_0. \end{aligned}$$

And all other coefficients B_n are zero.

From the initial velocity (p. 10) we get

$$\begin{aligned} u_t(x,0) &= \sum_{n=1}^{\infty} B_n^* \lambda_n \sin(n\pi x) = 0 \\ \implies \quad B_n^* &= 0 \text{ for all } n. \end{aligned}$$

Hence, the particular solution is

$$\boxed{u(x,t) = k_0 \cos(3\pi t) \sin(3\pi x)}$$ RESULT

Problem 5.5

Find the solution of the one-dimensional wave equation

$$u_{tt}(x,t) = u_{xx}(x,t), 0 \leq x \leq 1$$

subject to the boundary conditions

$$u(0,t) = u(1,t) = 0, \quad \text{for all } t$$

and the initial condition

$$u(x,0) = k_0 \sin(3\pi x), \quad u_t(x,0) = 3\pi k_0 \sin(3\pi x), 0 \leq x \leq 1,$$

where k_0 is a constant. You may start your calculation directly from the general solution given in the Lecture.

Derivation

As the equation, solution and initial elongation are the same as for Problem 5.4 we know that $B_3 = k_0$. We therefore start with the initial velocity (which differs in this case). We get

$$\begin{aligned} u_t(x,0) &= \sum_{n=1}^{\infty} B_n^* \lambda_n \sin(n\pi x) \\ &= 3\pi k_0 \sin(3\pi x) \\ \implies \quad B_3^* &= k_0. \end{aligned}$$

And all other coefficients $B_n^* = 0$. Hence,

$$\boxed{u(x,t) = k_0 \sin(3\pi x)(\cos(3\pi t) + \sin(3\pi t))} \qquad \text{RESULT}$$

Lecture 6: Heat equation

3. Oktober 2025

Problem 6.1

Consider the one-dimensional heat equation

$$u_t(x,t) = u_{xx}(x,t), 0 \le x \le 1$$

with boundary conditions

$$u(0,t) = u(1,t) = 0$$

for all t. Suppose you have the initial temperature profiles

$$u(x,0) = \sin(\pi x)$$

and the approximation (reflecting the situation where you do not know all details of the initial temperature profile)

$$u_N(x,0) = \sin(\pi x) + \sin(N\pi x)$$

where N is an integer number. Let $u(x,t)$ and $u_N(x,t)$ be the corresponding solutions to the above heat equation. Find the integrated squared error between these solutions (integration with respect to $0 \le x \le 1$) as a function of time.

Derivation

From the Lecture (p. 4) we have that

$$u(x,t) = \sum_{n=1}^{\infty} B_n e^{-\lambda_n^2 t} \sin\left(\frac{n\pi x}{L}\right)$$

with

$$\lambda_n = \frac{c\pi n}{L}, \quad B_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x.$$

Since the PDE is $u_t(x,t) = u_{xx}(x,t)$, $c^2 = 1$. Also, $L = 1$.

For our initial temperature profile our Fourier sine transform is simply sin with $B_1 = 1$ and all other $B_n = 0$. Thus

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} B_n e^{-\lambda_n^2 t} \sin(n\pi x) \\ &= e^{-\lambda_1^2 t} \sin(\pi x). \end{aligned}$$

The second initial temperature profile we a Fourier sine transform of sin with $B_1 = 1$, $B_N = 1$ and all other $B_n = 0$.

Hence

$$u(x,t) = e^{-\lambda_1^2 t} \sin(\pi x) + e^{-\lambda_N^2 t} \sin(N\pi x).$$

The integrated square error is defined as

$$E_N = \int_0^L \left(u(x,t) - u_N(x,t)\right)^2 \mathrm{d}x.$$

Hence

$$E_N = \int_0^1 \left(e^{-\lambda_1^2 t} \sin(\pi x) - e^{-\lambda_1^2 t} \sin(\pi x) - e^{-\lambda_N^2 t} \sin(N\pi x)\right)^2 \mathrm{d}x$$

$$= e^{-\lambda_N^2 t} \int_0^1 \sin^2(N\pi x)\,\mathrm{d}x$$
$$= e^{-\lambda_N^2 t} \left(\frac{1}{2} - \frac{\sin(2\pi N)}{4\pi N}\right).$$

$$\boxed{E_N = e^{-\lambda_N^2 t} \left(\frac{1}{2} - \frac{\sin(2\pi N)}{4\pi N}\right)}$$ RESULT

Problem 6.2

Consider the one-dimensional heat equation

$$u_t(x,t) = u_{xx}(x,t), 0 \le x \le L$$

with boundary conditions

$$u_x(0,t) = u_x(L,t) = 0$$

for all t and initial condition

$$u_t(x,0) = h(x), 0 \le x \le L.$$

Find the general solution for this situation.

Derivation

From the Lecture (p. 7) the general solution that fulfills the boundary conditions is

$$u(x,t) = B_0 + \sum_{n=1}^{\infty} B_n e^{-\lambda_n^2 t} \cos\left(\frac{n\pi x}{L}\right).$$

The time derivative of the above is

$$u_t(x,t) = \sum_{n=1}^{\infty} (-\lambda_n^2 B_n) e^{-\lambda_n^2 t} \cos\left(\frac{n\pi x}{L}\right).$$

Evaluating at $t = 0$ gives

$$u_t(x,0) = \sum_{n=1}^{\infty} \left(-\lambda_n^2 B_n\right) \cos\left(\frac{n\pi x}{L}\right) = h(x).$$

Now we have a Fourier cosine series for $h(x)$. Comparing this with the general representation:

$$u(x,0) = f(x) = B_0^* + \sum_{n=1}^{\infty} B_n^* \cos\left(\frac{n\pi x}{L}\right).$$

We see that $f(x) = h(x)$ when $B_0^* = 0$ and $B_n^* = -\lambda^2 B_n$.

The general formula for B_n^* is (p. 7)

$$B_n^* = \frac{2}{L} \int_0^L h(x) \cos\left(\frac{n\pi x}{L}\right) \mathrm{d}x.$$

Inserting our expression for B_n^* and solving for B_n we get

$$-\lambda^2 B_n = \frac{2}{L} \int_0^L h(x) \cos\left(\frac{n\pi x}{L}\right)$$

$$B_n = -\frac{2}{\lambda^2 L} \int_0^L h(x) \cos\left(\frac{n\pi x}{L}\right).$$

Which corresponds to the solution

$$\boxed{u(x,t) = B_0 + \sum_{n=1}^{\infty} \left(-\frac{2}{\lambda^2 L} \int_0^L h(x) \cos\left(\frac{n\pi x}{L}\right) \mathrm{d}x\right) e^{-\lambda_n^2 t} \cos\left(\frac{n\pi x}{L}\right)}$$ RESULT

Problem 6.3

Consider the PDE

$$u_{xx}(x,t) = u_{tt}(x,t), 0 \leq x \leq 1.$$

(a) Find the general solution that fulfils the conditions

$$u_x(0,t) = u_x(1,t) = 0$$

for all t.

(b) Find the general solution that fulfils the conditions

$$u_x(0,t) = u_x(1,t) = 0$$

for all t and

$$u(x,0) = 1 + \cos(3\pi x)$$

for $0 \leq x \leq 1$.

Derivation

a) We use the method of separation of variables.

$$u(x,t) = F(x)G(t).$$

and get

$$F''(x)G(t) = F(x)G''(t) \implies \frac{F''(x)}{F(x)} = \frac{G''(t)}{G(t)} = k.$$

We are therefore left with two ODE's, one for each of the unknown functions.

$$F''(x) - F(x)k = 0$$
$$G''(t) - G(t)k = 0$$

with the boundary conditions

$$F'(0)G(t) = 0 \implies F'(0) = 0$$
$$F'(1)G(t) = 0 \implies F'(1) = 0.$$

We start by assuming $k = 0$ (p. 3). For this case the spatial ODE is

$$F''(x) = 0 \implies F(x) = Ax + B$$

with some constants A and B.
Our boundary conditions are on u_x:

$$u_x(x,t) = F'(x)G(t)$$

so

$$u_x(0,t) = 0 \implies F'(0)G(t) = 0$$
$$u_x(1,t) = 0 \implies F'(1)G(t) = 0.$$

Here we want a non-trivial solution, i.e. $G(t) \neq 0$. This forces

$$F'(0) = 0, \quad F'(1) = 0$$

but since $F'(x) = A$ these conditions just give $A = 0$. Therefore $F(x) = B \quad \text{(constant)}$. We choose $B = 1$ and get the interesting case $F(0) = 1$.
If we instead assume $k > 0$ (ibid.) we then get

$$F_{xx}(x) = kF(x) \implies F(x) = Ae^{\sqrt{k}x} + Be^{-\sqrt{k}x}$$

for some constants A and B. To apply the boundary conditions we take the derivative of this and get

$$F'(x) = A\sqrt{k}e^{\sqrt{k}x} - B\sqrt{k}e^{-\sqrt{k}x}.$$

We now apply the boundary conditions as

$$\begin{aligned} F'(0) &= 0 \\ A\sqrt{k} - B\sqrt{k} &= 0 \\ \implies A &= B \\ \implies A\sqrt{k}e^{\sqrt{k}x} - B\sqrt{k}e^{-\sqrt{k}x} &= A\sqrt{k}\left(e^{\sqrt{k}x} - e^{-\sqrt{k}x}\right) \\ F'(1) &= 0 \\ \implies A &= 0 = B. \end{aligned}$$

Hence, we only get the uninteresting case $F(x) = 0$ for $k > 0$.
For $k < 0$ (p. 6) we assume $p = \sqrt{-k}$ and get

$$F_{xx}(x) + p^2F(x) = 0 \implies F(x) = A\cos(px) + B\sin(px)$$

for some constants A and B. We take the derivative of this to get

$$F'(x) = -Ap\sin(px) + Bp\cos(px).$$

Applying the boundary conditions we get

$$\begin{aligned} F'(0) &= 0 \\ Bp\cos p &= 0 \\ \implies B &= 0 \\ F'(1) &= 0 \\ -Ap\sin(p) &= 0 \\ \implies p &= p_n = n\pi, \quad n = 1, 2, 3, \ldots. \end{aligned}$$

We choose $A = 1$ and for each $n = 0, 1, 2, 3, \ldots$ the solution becomes (p. 7)

$$F_n(x) = \cos(n\pi x), \quad n = 0, 1, 2, 3, \ldots.$$

We therefore have the two interesting cases $F_0(x) = 1$ and $F_n = \cos(n\pi x)$.
For the temporal equation we start assuming $k = 0$ as an interesting case was found here. We get

$$\begin{aligned} G''(t) &= 0 \\ G'(t) &= B_0^* \\ G_0(t) &= B_0 + B_0^* t. \end{aligned}$$

For $k < 0$, where the other interesting case was found we get

$$G''(t) - kG(t) = 0$$

$$G''(t) - \left(-p_n^2\right) G(t) = 0$$
$$G_n(t) = B_n \cos(n\pi t) + B_n^* \sin(n\pi t).$$

We therefore have the two fundamental solutions

$$u_0(x,t) = G_0(t)F_0(x) = B_0 + B_0^* t$$
$$u_n(x,t) = G_n(t)F_n(x) = \left(B_n \cos(n\pi t) + B_n^* \sin(n\pi t)\right)\cos(n\pi x)$$
$$.$$

Which gives the general solution

$$u(x,t) = u_0(x,t) + \sum_{n=1}^{\infty} u_n(x,t) = B_0 + B_0^* t + \sum_{n=1}^{\infty} \left(B_n \cos(n\pi t) + B_n^* \sin(n\pi t)\right)\cos(n\pi x).$$

b) Here, we have the additional condition that $u(x,0) = 1 + \cos(3\pi x)$ for $0 \le x \le 1$. Writing this as a Fourier cosine series

$$u(x,0) = B_0 + \sum_{n=1}^{\infty} B_n \cos(n\pi x) = 1 + \cos(3\pi x).$$

We see that $B_0 = 1$ and $B_3 = 1$ and all other coefficients $B_n = 0$.
Hence the general solution is

$$u(x,t) = 1 + B_0^* t + \cos(3\pi t)\cos(3\pi x) + \sum_{n=1}^{\infty} B_n^* \sin(n\pi t)\cos(n\pi x).$$

a) $u(x,t) = B_0 + B_0^* t + \sum_{n=1}^{\infty} \left(B_n \cos(n\pi t) + B_n^* \sin(n\pi t)\right)\cos(n\pi x)$

b) $u(x,t) = 1 + B_0^* t \cos(3\pi t)\cos(3\pi x) + \sum_{n=1}^{\infty} B_n^* \sin(n\pi t)\cos(n\pi x).$

RESULT

Problem 6.4

Consider the one-dimensional heat equation

$$u_t(x,t) = u_{xx}(x,t), 0 \le x \le L$$

with boundary conditions

$$u(0,t) = u(L,t) = 0.$$

(a) Calculate the average temperature

$$\frac{1}{L}\int_0^L u(x,t)\,\mathrm{d}x$$

for the initial condition

$$u(x,0) = \sin\left(\frac{\pi x}{L}\right).$$

(b) Calculate the average temperature for the initial condition

$$u(x,0) = \sin\left(\frac{\pi x}{L}\right) + \sin\left(\frac{2\pi x}{L}\right).$$

Derivation

Note: Relevant theory on p. 2

a) From the Lecture (pp. 4-5) we know that for an initial temperature profile $u(x,0)$ expressed as a sum of temperature waves each will decay exponentially with time as

$$e^{-\lambda_n^2 t}.$$

Hence, for the initial condition $u(x,0) = \sin(\pi x/L)$ the solution is

$$u(x,t) = e^{-\lambda_1^2 t}\sin\left(\frac{x\pi}{L}\right).$$

Plugging this into the equation we get

$$\frac{1}{L}\int_0^L e^{-\lambda_1^2 t}\sin\left(\frac{\pi x}{L}\right)\mathrm{d}x = \frac{2e^{-\lambda_1^2 t}}{\pi}.$$

b) Here, we get the solution

$$u(x,t) = e^{-\lambda_1^2 t}\sin\left(\frac{\pi x}{L}\right) + e^{-\lambda_2^2 t}\sin\left(\frac{2\pi x}{L}\right).$$

Plugging this into the equation we get

$$\frac{1}{L}\int_0^L e^{-\lambda_1^2 t}\sin\left(\frac{\pi x}{L}\right) + e^{-\lambda_2^2 t}\sin\left(\frac{2\pi x}{L}\right)\mathrm{d}x = \frac{2e^{-\lambda_1^2 t}}{\pi}.$$

a) $\dfrac{2e^{-\lambda_1^2 t}}{\pi}$

b) $\dfrac{2e^{-\lambda_1^2 t}}{\pi}.$

RESULT

Lecture 7: The heat equation continued

24. Oktober 2025

Problem 7.1

Consider the one-dimensional heat equation

$$u_t(x,t) = c^2 u_{xx}(x,t), 0 \le x \le L$$

subject to the boundary conditions

$$u(0,t) = u(L,t) = 0$$

for all t.

Let T_0 be a constant and

$$u(x,0) = \begin{cases} T_0, & 0 < x < L \\ 0, & x = 0 \\ 0, & x = L. \end{cases}$$

This set-up can be used to describe the situation where a very thin, latterally insulated bar of constant temperature T_0 is, at time $t = 0$, thrown into a heat bath of zero temperature. Find the solution $u(x,t)$.

Derivation

From the Lecture (p. 2) we have the general solution

$$u(x,t) = \sum_{n=1}^{\infty} B_n e^{-\lambda_n^2 t} \sin\left(\frac{n\pi x}{L}\right).$$

From the Lecture (p. 3) the coefficients B_n are determined as

$$B_n = \frac{2}{L}\int_0^L (f(x)) \sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x.$$

Substituting our variables we get

$$\begin{aligned} B_n &= \frac{2}{L}\int_0^L T_0 \sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x \\ &= \frac{2T_0}{L}\int_0^L \sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x \\ &= \frac{2T_0(1-\cos(\pi n))}{\pi n}. \end{aligned}$$

Hence, the solution is

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} \frac{2T_0(1-\cos(\pi n))}{\pi n} e^{-\lambda_n^2 t} \sin\left(\frac{n\pi x}{L}\right)}$$ RESULT

Problem 7.2

Consider the one-dimensional heat equation

$$u_t(x,t) = c^2 u_{xx}(x,t), 0 \le x \le L$$

subject to the boundary conditions

$$u(0,t) = T_1, u(L,t) = T_2$$

for all t.

Let T_0 be a constant and

$$u(x,0) = \begin{cases} \frac{T_2 - T_1}{L} x + T_0, & 0 < x < L \\ T_1, & X = 0 \\ T_2, & x = L. \end{cases}$$

Find the solution $u(x,t)$.

Derivation

From the Lecture (p. 2) we have the general solution

$$u(x,t) = \frac{T_2 - T_1}{L} x + T_1 + \sum_{n=1}^{\infty} B_n e^{-\lambda_n^2 t} \sin\left(\frac{n\pi x}{L}\right)$$

with coefficients

$$B_n = \frac{2}{L} \int_0^L \left(f(x) - \frac{T_2 - T_1}{L} x - T_1 \right) \sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x.$$

Substituting, we get

$$\begin{aligned} B_n &= \frac{2}{L} \int_0^L \left(\frac{T_2 - T_1}{L} x + T_0 - \frac{T_2 - T_1}{L} x - T_1 \right) \sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x \\ &= \frac{2}{L} \int_0^L (T_0 - T_1) \sin\left(\frac{n\pi x}{L}\right) \\ &= \frac{2(T_0 - T_1)(1 - \cos(\pi n))}{\pi n}. \end{aligned}$$

Which gives the solution

$$\boxed{u(x,t) = \frac{T_2 - T_1}{L} x + T_1 + \sum_{n=1}^{\infty} \frac{2(T_0 - T_1)(1 - \cos(\pi n))}{\pi n} e^{-\lambda_n^2 t} \sin\left(\frac{n\pi x}{L}\right)}$$

RESULT

Problem 7.3

Consider the steady heat flow on a thin plate in the xy-plane, $0 \le x \le 1, 0 \le y \le 1$, insulated from above/below, governed by

$$u_{xx}(x,y) + u_{yy}(x,y) = 0.$$

Find the solution that corresponds to the boundary conditions

$$\begin{aligned} &u(x,0) = 0, && u(x,1) = 0, && 0 \le x \le 1 \\ &u(0,y) = 0, && u(1,y) = 0, && 0 \le y \le 1. \end{aligned}$$

Derivation

From the Lecture (p. 7) we have the general solution

$$u(x,y) = \sum_{n=1}^{\infty} C_n^* \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi y}{a}\right)$$

where $a = 1$ is the length of the domain in the x direction. Hence,

$$u(x,y) = \sum_{n=1}^{\infty} C_n^* \sin(n\pi x) \sinh(n\pi y).$$

We use the upper boundary condition and get

$$\begin{aligned} u(x,1) &= 0 \\ &= \sum_{n=1}^{\infty} C_n^* \sin(n\pi x) \sinh(n\pi) \\ &= C_n^* = 0 \end{aligned}$$

for all n.

The unique solution is therefore

$$\boxed{u(x,y) = 0}$$ RESULT

Problem 7.4

Consider the steady heat flow on a thin plate in the xy-plane, $0 \le x \le 3, 0 \le y \le 2$, insulated from above/below, governed by

$$u_{xx}(x,y) + u_{yy}(x,y) = 0.$$

Find the solution that corresponds to the boundary conditions

$$\begin{aligned} &u(x,0) = 0, && u(x,2) = 2\sin\frac{4\pi x}{3}, && 0 \le x \le 3 \\ &u(0,y) = 0, && u(3,y) = 0, && 0 \le y \le 2. \end{aligned}$$

Derivation

Here, we have the general solution (p. 7)

$$u(x,y) = \sum_{n=1}^{\infty} C_n^* \sin\left(\frac{n\pi x}{3}\right)\sinh\left(\frac{n\pi y}{3}\right).$$

We use the upper boundary condition and get

$$\begin{aligned} u(x,2) &= 2\sin\frac{8\pi}{3} \\ 2\sin\frac{8\pi}{3} &= \sum_{n=1}^{\infty} C_n^* \sin\left(\frac{2n\pi}{3}\right)\sinh\left(\frac{2n\pi}{3}\right) \end{aligned}$$

by comparison, we see that

$$C_4^* \sinh\frac{8\pi}{3} = 2 \implies C_4^* = \frac{2}{\sinh\frac{8\pi}{3}}.$$

Hence, the solution is

$$\boxed{u(x,y) = \frac{2}{\sinh\frac{8\pi}{3}}\sin\left(\frac{4\pi x}{3}\right)\sinh\left(\frac{4\pi y}{3}\right)}$$ RESULT

Problem 7.5

Consider the steady heat flow on a thin plate in the xy-plane, $0 \le x \le 1, 0 \le y \le 1$, insulated from above/below, governed by

$$u_{xx}(x,y) + u_{yy}(x,y) = 0.$$

Find the solution that corresponds to the boundary conditions

$$\begin{aligned} &u(x,0) = 0, \qquad && u_y(x,1) = 0, \qquad && 0 \le x \le 1 \\ &u(0,y) = 0, \qquad && u(1,y) = 0, \qquad && 0 \le y \le 1. \end{aligned}$$

Derivation

See Problem 7.3.

Lecture 8: Vibrating rectangular membranes and Double Fourier Sine Series 31. Oktober 2025

Problem 8.1

Consider the function

$$f(x,y) = 3\sin\left(\frac{4\pi y}{3}\right)\sin\left(\frac{3\pi x}{2}\right) + 2\sin\left(\frac{4\pi y}{3}\right)\sin\left(\frac{3\pi x}{6}\right), \qquad 0 \le x \le 2, 0 \le y \le 3.$$

Find the Fourier coefficients of the double Fourier sine series of $f(x,y)$.

Hint: Coefficients of double Fourier sine series are unique.

Derivation

From the Lecture (p. 2) the general representation of the double Fourier sine series of $f(x,y)$ is

$$f(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} S_{m,n}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right).$$

First of all we simplify the given function as

$$f(x,y) = 3\sin\left(\frac{3\pi x}{2}\right)\sin\left(\frac{4\pi y}{3}\right) + 2\sin\left(\frac{\pi x}{2}\right)\sin\left(\frac{4\pi y}{3}\right).$$

By comparison, we see that

$$\boxed{S_{3,4} = 3, \quad S_{1,4} = 2, \qquad \text{All other Fourier coefficients are } 0} \qquad \textsc{Result}$$

Problem 8.2

Consider the deflection $u(x,t)$ of a vibrating elastic membrane in the xy-plane, $0 \le x \le 1, 0 \le y \le 1$, fixed along its boundaries and governed by

$$u_{tt}(x,y,t) = c^2 \left(u_{xx}(x,y,t) + u_{yy}(x,y,t)\right).$$

(a) Find the solution that corresponds to the initial conditions

$$\begin{aligned} u(x,y,0) &= \sin(6\pi x)\sin(2\pi y) \\ u_t(x,y,0) &= 0. \end{aligned}$$

(b) Find the solution that corresponds to the initial conditions

$$\begin{aligned} u(x,y,0) &= 0 \\ u_t(x,y,0) &= \sin(6\pi x)\sin(2\pi y). \end{aligned}$$

(c) Find the solution that corresponds to the initial conditions

$$\begin{aligned} u(x,y,0) &= \sin(6\pi x)\sin(2\pi y) \\ u_t(x,y,0) &= \sin(6\pi x)\sin(2\pi y). \end{aligned}$$

Hint: Coefficients of double Fourier dine series are unique.

Derivation

a) From the Lecture (p. 6) the general solution is

$$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(E_{m,n}\cos(\lambda_{m,n}t) + J_{m,n}\sin(\lambda_{m,n}t)\right)\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right).$$

Which, in this case simplifies to

$$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(E_{m,n}\cos(\lambda_{m,n}t) + J_{m,n}\sin(\lambda_{m,n}t)\right)\sin(m\pi x)\sin(n\pi y).$$

We now employ the initial elongation (p. 6) to get

$$\begin{aligned} u(x,y,0) &= \sin(6\pi x)\sin(2\pi y) \\ &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} E_{m,n}\sin(n\pi x)\sin(m\pi y) \\ \implies E_{6,2} &= 1. \end{aligned}$$

Using the initial velocity we get

$$\begin{aligned} u_t(x,y,0) &= 0 \\ &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(J_{m,n}\lambda_{m,n}\cos(\lambda_{m,n}t)\right)\sin(m\pi x)\sin(n\pi y) \\ \implies J_{m,n} &= 0. \end{aligned}$$

From the Lecture (p. 5) we have that

$$\lambda_{m,n} = c\pi\sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}.$$

To find $\lambda_{6,2}$ we simply use

$$\lambda_{6,2} = c\pi\sqrt{\frac{6^2}{1} + \frac{2^2}{1}} = c\pi\sqrt{40}.$$

Hence, the solution is

$$u(x, y, t) = \cos\left(c\pi\sqrt{40}t\right)\sin(6\pi x)\sin(2\pi y).$$

b) In this gave the initial elongation gives

$$u(x, y, 0) = 0 \implies E_{m,n} = 0.$$

From the initial velocity, we get

$$\begin{aligned} u_t(x, y, 0) &= \sin(6\pi x)\sin(2\pi y) \\ &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(J_{m,n}\lambda_{m,n}\right)\sin(m\pi x)\sin(n\pi y) \\ \implies J_{6,2}\lambda_{6,2} &= 1 \\ J_{6,2} &= \frac{1}{c\pi\sqrt{40}}. \end{aligned}$$

Hence, the solution is

$$u(x, y, t) = \frac{1}{c\pi\sqrt{40}}\sin\left(c\pi\sqrt{40}t\right)\sin(6\pi x)\sin(2\pi y).$$

c) For this case we have in the above two parts found $E_{6,2} = 1$ and $J_{6,2} = 1/(\sqrt{40}c\pi)$. Hence, the general solution is

$$u(x, y, t) = \left(\cos(\sqrt{40}c\pi t) + \frac{1}{\sqrt{40}c\pi}\sin(\sqrt{40}c\pi t)\right)\sin(6\pi x)\sin(2\pi y).$$

a) $u(x, y, t) = \cos\left(\sqrt{40}c\pi t\right)\sin(6\pi x)\sin(2\pi y)$

b) $u(x, y, t) = \dfrac{1}{\sqrt{40}c\pi}\sin\left(\sqrt{40}c\pi t\right)\sin(6\pi x)\sin(2\pi y)$

c) $u(x, y, t) = \left(\cos\left(\sqrt{40}c\pi t\right) + \dfrac{1}{\sqrt{40}c\pi}\sin\left(\sqrt{40}c\pi t\right)\right)\sin(6\pi x)\sin\left(2\pi y\right).$

RESULT

Problem 8.3

Consider the two-dimensional heat equation

$$u_t(x, y, t) = c^2(u_{xx}(x, y, t) + u_{yy}(x, y, t)).$$

(a) Assume $u(x, y, t) = F(x, y)G(t)$ and derive a PDE for $F(x, y)$ and an ODE for $G(t)$.

(b) Assume $F(x, y) = H(x)Q(y)$ to finally derive three ODEs for $H(x)$, $Q(y)$ and $G(t)$.

Derivation

a) By insertion, we obtain

$$\begin{aligned}
F(x,y)G'(t) &= c^2\left(F_{xx}(x,y)G(t) + F_{yy}(x,y)G(t)\right)\\
\frac{G'(t)}{c^2 G(t)} &= \frac{F_{xx}(x,y) + F_{yy}(x,y)}{F(x,y)} = c_0\\
\implies F_{xx}(x,y) + F_{yy}(x,y) - c_0 F(x,y) &= 0\\
G'(t) - c_0 c^2 G(t) &= 0.
\end{aligned}$$

b) Again, we insert and get

$$\begin{aligned}
H''(x)Q(y) + H(x)Q''(y) - c_0 H(x)Q(y) &= 0\\
\frac{H''(x)}{H(x)} &= -\frac{Q''(y)}{Q(y)} + c_0 = c_1\\
\implies H''(x) - c_1 H(x) &= 0\\
Q''(y) - (c_0 - c_1)Q(y) &= 0\\
G'(t) - c_0 c^2 G(t) &= 0.
\end{aligned}$$

$$\boxed{\begin{aligned}
H''(x) - c_1 H(x) &= 0\\
Q''(y) - (c_0 - c_1)Q(y) &= 0\\
G'(t) - c_0 c^2 G(t) &= 0.
\end{aligned}}$$

RESULT

Problem 8.4

Consider the PDE

$$u_t(x,y,z,t) = u_x(x,y,z,t) + u_y(x,y,z,t) + u_z(x,y,z,t).$$

Assume $u(x,y,z,t) = H(x)Q(y)P(z)G(t)$ and derive 4 ODEs for $H(x), Q(y), P(z)$, and $G(t)$.

Derivation

We assume $u(x,y,z,t) = A(x,y,z)G(t)$ and get

$$\begin{aligned}
A(x,y,z)G'(t) &= A_x(x,y,z)G(t) + A_y(x,y,z)G(t) + A_z(x,y,z)G(t) \\
\frac{G'(t)}{G(t)} &= \frac{A_x(x,y,z) + A_y(x,y,z) + A_z(x,y,z)}{A(x,y,z)} = c_0 \\
\implies G'(t) - c_0 G(t) &= 0.
\end{aligned}$$

Leaving us with

$$A_x(x,y,z) + A_y(x,y,z) + A_z(x,y,z) - c_0 A(x,y,z) = 0.$$

Now we assume $A(x,y,z) = B(x,y)P(z)$ and get

$$\begin{aligned}
B_x(x,y)P(z) + B_y(x,y)P(z) + B(x,y)P'(z) - c_0 B(x,y)P(z) = 0 \\
B_x(x,y)P(z) + B_y(x,y)P(z) &= c_0 B(x,y)P(z) - B(x,y)P'(z) \\
\frac{B_x(x,y) + B_y(x,y)}{B(x,y)} &= c_0 - \frac{P'(z)}{P(z)} = c_1 \\
\implies P'(z) - (c_0 - c_1)P(z) &= 0.
\end{aligned}$$

Leaving us with

$$B_x(x,y) + B_y(x,y) - c_1 B(x,y) = 0.$$

Now we assume $B(x,y) = H(x)Q(y)$ and get

$$\begin{aligned}
H'(x)Q(y) + H(x)Q'(y) - c_1 H(x)Q(y) = 0 \\
\frac{H'(x)}{H(x)} &= c_1 - \frac{Q'(y)}{Q(y)} = c_2 \\
H'(x) - c_2 H(x) = 0 \\
Q'(y) - (c_1 - c_2)Q(y) = 0.
\end{aligned}$$

$$\boxed{\begin{aligned}
H'(x) - c_2 H(x) &= 0 \\
Q'(y) - (c_1 - c_2)Q(y) &= 0 \\
P'(z) - (c_0 - c_1)P(z) &= 0 \\
G'(t) - c_0 G(t) &= 0.
\end{aligned}}$$

RESULT

Lecture 9: D'Alembert's solution of the one-dimensional wave equation 7. November 2025

Problem 9.1

Calculate the second derivatives of

$$u(x,t) = \Phi(x+ct) + \Psi(x-ct)$$

with respect to t and x and show that any function of this type is a solution to the one-dimensional wave equation

$$u_{tt}(x,t) = c^2 u_{xx}(x,t)$$

for arbitrary twice differentiable functions Φ and Ψ.

Derivation

We start by calculating the second derivative with respect to t, remembering to use the chain rule as

$$\begin{aligned}
\frac{\partial u(x,t)}{\partial t} &= \frac{\mathrm{d}\Phi(x+ct)}{\mathrm{d}t} + \frac{\mathrm{d}\Psi(x-ct)}{\mathrm{d}t} \\
&= \frac{\mathrm{d}\Phi(x+ct)}{\mathrm{d}(x+ct)}\frac{\mathrm{d}(x+ct)}{\mathrm{d}t} + \frac{\mathrm{d}\Psi(x-ct)}{\mathrm{d}(x-ct)}\frac{\mathrm{d}(x-ct)}{\mathrm{d}t} \\
&= c\left(\frac{\mathrm{d}\Phi(x+ct)}{\mathrm{d}(x+ct)} - \frac{\mathrm{d}\Psi(x-ct)}{\mathrm{d}(x-ct)}\right) \\
\frac{\partial^2 u(x,t)}{\partial t^2} &= c\left(\frac{\mathrm{d}^2\Phi(x+ct)}{\mathrm{d}(x+ct)^2}\frac{\mathrm{d}(x+ct)}{\mathrm{d}t} - \frac{\mathrm{d}^2\Phi(x-ct)}{\mathrm{d}(x-ct)^2}\frac{\mathrm{d}(x-ct)}{\mathrm{d}t}\right) \\
&= c^2\left(\frac{\mathrm{d}^2\Phi(x+ct)}{\mathrm{d}(x+ct)^2} + \frac{\mathrm{d}^2\Psi(x-ct)}{\mathrm{d}(x-ct)^2}\right)
\end{aligned}$$

.

Now we calculate the second derivatives with respect to x and get

$$\begin{aligned}
\frac{\partial u(x,t)}{\partial x} &= \frac{\mathrm{d}\Phi(x+ct)}{\mathrm{d}x} + \frac{\mathrm{d}\Psi(x-ct)}{\mathrm{d}x} \\
&= \frac{\mathrm{d}\Phi(x+ct)}{\mathrm{d}(x+ct)}\frac{\mathrm{d}(x+ct)}{\mathrm{d}x} + \frac{\mathrm{d}\Psi(x-ct)}{\mathrm{d}(x-ct)}\frac{\mathrm{d}(x-ct)}{\mathrm{d}x} \\
&= \frac{\mathrm{d}\Phi(x+ct)}{\mathrm{d}(x+ct)} - \frac{\mathrm{d}\Psi(x-ct)}{\mathrm{d}(x-ct)} \\
\frac{\partial^2 u(x,t)}{\partial t^2} &= \frac{\mathrm{d}^2\Phi(x+ct)}{\mathrm{d}(x+ct)^2}\frac{\mathrm{d}(x+ct)}{\mathrm{d}x} - \frac{\mathrm{d}^2\Phi(x-ct)}{\mathrm{d}(x-ct)^2}\frac{\mathrm{d}(x-ct)}{\mathrm{d}x} \\
&= \frac{\mathrm{d}^2\Phi(x+ct)}{\mathrm{d}(x+ct)^2} + \frac{\mathrm{d}^2\Psi(x-ct)}{\mathrm{d}(x-ct)^2}
\end{aligned}$$

.

Since

$$\begin{aligned}
\frac{\partial^2 u(x,t)}{\partial t^2} &= c^2\frac{\partial^2 u(x,t)}{\partial x^2} \\
\frac{\mathrm{d}^2\Phi(x+ct)}{\mathrm{d}(x+ct)^2} + \frac{\mathrm{d}^2\Phi(x-ct)}{\mathrm{d}(x-ct)} &= c^2\left(\frac{\mathrm{d}^2\Phi(x+ct)}{\mathrm{d}(x+ct)^2} + \frac{\mathrm{d}^2\Psi(x-ct)}{\mathrm{d}(x-ct)^2}\right)
\end{aligned}$$

is is shown.

Problem 9.2

Consider the PDE

$$u_{xt}(x,t) = 0$$

and the change of variables

$$\begin{aligned} v &= v(x,t) = x + 2t, & w &= w(x,t) = x - 2t \\ x &= x(v,w) = \frac{1}{2}(v+w), & t &= t(v,w) = \frac{1}{4}(v-w). \end{aligned}$$

Derive the corresponding PDE for

$$\overline{u}(v,w) = u(x(v,w), t(v,w)).$$

You may assume that $\overline{u}(v,w)$ is twice continuously differentiable, i.e. $\overline{u}_{vw}(v,w) = \overline{u}_{wv}(v,w)$.

Derivation

From the Lecture (p. 2), we have that

$$u_{xt}(x,t) = \overline{u}_{xt}(v(x,t), w(x,t)).$$

We take the first derivative in t, using the chain rule, and get

$$\begin{aligned} \overline{u}_t(x,t) &= \overline{u}_v(v,w)\, v_t(x,t) + \overline{u}_w(v,w)\, w_t(x,t) \\ &= \overline{u}_v(v,w) \cdot 2 + \overline{u}_w(v,w) \cdot (-2) \\ &= 2\left(\overline{u}_v(v,w) - \overline{u}_w(v,w)\right). \end{aligned}$$

We now take the derivative of this in x to get

$$\begin{aligned} \overline{u}_{xt}(x,t) &= 2\left(\overline{u}_{vv}(v,w)\, v_x(x,t) + \overline{u}_{wv}(v,w)\, w_x(x,t) - \overline{u}_{uw}(v,w)\, v_x(x,t) - \overline{u}_{ww}(v,w)\, w_x(x,t)\right) \\ &= 2\left(\overline{u}_{vv}(v,w) - \overline{u}_{ww}(v,w)\right). \end{aligned}$$

Now, we insert this into the original PDE:

$$\begin{aligned} u_{xt}(x,t) &= 0 \\ 2\left(\overline{u}_{vv}(v,w) - \overline{u}_{ww}(v,w)\right) &= 0 \\ \overline{u}_{vv}(v,w) - \overline{u}_{ww}(v,w) &= 0. \end{aligned}$$

$$\boxed{\overline{u}_{vv}(v,w) - \overline{u}_{ww}(v,w) = 0}$$ RESULT

Problem 9.3

Calculate d'Alembert's solution corresponding to the initial conditions

$$u(x,0) = k_0 \sin(3\pi x), \quad u_t(x,0) = 3\pi k_0 \sin(3\pi x), \quad -\infty < x < \infty,$$

where k_0 is a constant.

Compare your result to that of Exercice 5, Lecture 5.

Hint: $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta, \quad \cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta.$

Derivation

From the Lecture (p. 6) d'Alembert's solution to the one-dimensional wave equation is

$$u(x,t) = \frac{1}{2}\big(f(x+ct) + f(x-ct)\big) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,\mathrm{d}s$$

with initial conditions

$$\begin{aligned} u(x,0) &= f(x) = k_0 \sin(3\pi x), && -\infty < x < \infty \\ u_t(x,0) &= g(x) = 3\pi k_0 \sin(3\pi x), && -\infty < x < \infty. \end{aligned}$$

By insertion we get

$$\begin{aligned} u(x,t) &= \frac{1}{2}\left(k_0\sin(3\pi(x+ct)) + k_0\sin(3\pi(x-ct))\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} 3\pi k_0 \sin(3\pi s)\,\mathrm{d}s \\ &= \frac{k_0}{2}\left(\sin(3\pi x + 3\pi ct)) + \sin(3\pi x - 3\pi ct)\right) + \frac{3\pi k_0}{2c}\int_{x-ct}^{x+ct} \sin(3\pi s)\,\mathrm{d}s \\ &= \frac{k_0}{2}\left(\sin(3\pi x)\cos(3\pi ct) + \cos(3\pi x)\sin(3\pi ct) + \sin(3\pi x)\cos(-3\pi ct) + \cos(3\pi x)\sin(-3\pi ct)\right) \\ &\qquad + \frac{3\pi k_0}{2c}\cdot\frac{2\sin(3\pi x)\sin(3\pi ct)}{3\pi} \\ &= \frac{k_0}{2}\left(2\sin(3\pi x)\cos(3\pi ct)\right) + \frac{k_0}{c}\sin(3\pi x)\sin(3\pi ct) \\ &= k_0\sin(3\pi x)\cos(3\pi ct) + \frac{k_0}{c}\sin(3\pi x)\sin(3\pi ct). \end{aligned}$$

$$\boxed{u(x,t) = k_0\sin(3\pi x)\cos(3\pi ct) + \frac{k_0}{c}\sin(3\pi x)\sin(3\pi ct)}$$

RESULT

Lecture 10: Solutions of PDEs by Fourier methods

14. November 2025

Problem 10.1

Consider the one-dimensional heat equation and the Fourier transform of the solution $\hat{u}(\omega, t) = C(\omega)e^{-c^2\omega^2 t}$ derived in Section 7.1.

a) Use the modified initial condition

$$u_x(x, 0) = f(x).$$

to fint an expression relating $C(\omega)$ to the Fourier transform of f.

b) How does expression (1) in Section 7.1 change under the situation described in a)?

c) Use the result from b) and arguments similar to those used in the derivation of (2) in Section 7.1 to derive an expression for $u(x, t)$ that does not contain the complex unit i.

Derivation

a) From the Lecture (p. 1), we have the general solution:

$$\hat{u}(\omega, t) = C(\omega)e^{-c^2\omega^2 t}.$$

Following the same procedure as in Section 7.1, we start by taking the Fourier transform of the modified initial condition, $u_x(x, 0) = f(x)$ as

$$\begin{aligned} \mathscr{F}(u_x(x,t)) &= i\omega\mathscr{F}(u(x,t)) \\ \mathscr{F}(f) = \mathscr{F}(u_x(x,0)) &= i\omega\mathscr{F}(u(x,0)) \\ \mathscr{F}(u(x,0) &= \frac{1}{i\omega}\mathscr{F}(f). \end{aligned}$$

Furthermore, we have that $\mathscr{F}(u(x,0)) = \hat{u}(\omega, t)$. Hence,

$$\begin{aligned} \hat{u}(\omega, t) &= \frac{1}{i\omega}\mathscr{F}(f) = C(\omega)e^{-c^2\omega^2 t} \\ \implies \mathscr{F}(u(x,0)) &= C(\omega) \\ C(\omega) &= \frac{1}{i\omega}\mathscr{F}(f). \end{aligned}$$

b) The solution is therefore

$$\hat{u}(\omega, t) = \frac{1}{i\omega}\hat{f}(\omega)e^{-c^2\omega^2 t}.$$

Now, to get an expression like expression (1) we must take the inverse Fourier transform. We use:

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(v)e^{-iv\omega}\,\mathrm{d}v.$$

and get

$$\begin{aligned} u(x,t) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{u}(\omega,t)e^{ix\omega}\,\mathrm{d}\omega \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{1}{i\omega}\hat{f}(\omega)e^{-c^2\omega^2 t}e^{i\omega x}\,\mathrm{d}\omega \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\frac{1}{i\omega}\int_{-\infty}^{\infty} f(v)e^{-iv\omega}\,\mathrm{d}v\, e^{-c^2\omega^2 t}\,\mathrm{d}\omega \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(v)\left(\int_{-\infty}^{\infty} e^{-c^2\omega^2 t}\frac{e^{i(\omega x-\omega v)}}{i\omega}\,\mathrm{d}\omega\right)\mathrm{d}v. \end{aligned}$$

c) Using the Euler formula

$$e^{i\alpha} = \cos(\alpha) + i\sin(\alpha)$$

we can rewrite our inner integrand as:

$$\begin{aligned} e^{-c^2\omega^2 t}\frac{e^{i(\omega x-\omega v)}}{i\omega} &= e^{-c^2\omega^2 t} \\ &= e^{-c^2\omega^2 t}\frac{\cos(\omega(x-v))}{i\omega} + i e^{-c^2\omega^2 t}\frac{\sin(\omega(x-v))}{i\omega} \\ &= 2\int_0^\infty e^{-c^2\omega^2 t}\frac{\sin(\omega(x-v))}{\omega}\,\mathrm{d}\omega. \end{aligned}$$

Hence,

$$u(x,t) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\int_0^\infty e^{-c^2\omega^2 t}\frac{\sin\omega(x-v)}{\omega}\,\mathrm{d}\omega\,\mathrm{d}v.$$

a) $C(\omega) = \frac{1}{i\omega}\mathscr{F}(f)$

b) $u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(v)\int_{-\infty}^{\infty} e^{-c^2\omega^2 t}\frac{e^{i(\omega x-\omega v)}}{i\omega}\,\mathrm{d}\omega\,\mathrm{d}v$

c) $u(x,t) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\int_0^\infty e^{-c^2\omega^2 t}\frac{\sin(\omega(x-v))}{\omega}\,\mathrm{d}\omega\,\mathrm{d}v.$

RESULT

Problem 10.2

Use the Fourier cosine transform to find the solution of

$$u_t(x,t) = c^2 u_{xx}(x,t), \quad 0 \le x < \infty$$

subject to the initial condition

$$u(x,0) = f(x) \quad 0 \le x < \infty$$

and the boundary condition

$$u_x(0,t) = 0.$$

You may exchange integration and differentiation whenever needed.

Derivation

Following the logic from the section on Fourier sine transforms (p. 3) we start by taking the cosine transform of the heat equation and get

$$\begin{aligned}
\mathscr{F}_C(u_t(x,t)) &= \mathscr{F}_C\left(c^2 u_{xx}(x,t\right) \\
&= c^2 \mathscr{F}_C\left(u_{xx}(x,t)\right) \\
&= c^2\left(-\omega^2 \mathscr{F}_C(u(x,t)) - \sqrt{\frac{2}{\pi}} u_x(0,t)\right) \\
&= -c^2\omega^2 \mathscr{F}_C(u(x,t)) \\
&= -c^2\omega^2 \hat{u}_C(\omega,t) \\
\frac{\partial \hat{u}_C(\omega,t)}{\partial t} &= -c^2\omega^2 \hat{u}_C(\omega,t).
\end{aligned}$$

This has the general solution

$$\hat{u}_C(\omega,t) = C(\omega) e^{-c^2\omega^2 t}.$$

From the initial condition we get

$$\begin{aligned}
\mathscr{F}_C(u(x,0)) &= \mathscr{F}_C(f(x)) \\
\hat{u}_C(\omega,0) &= \hat{u}_C(f(\omega)) \\
\implies C(\omega) &= \hat{f}_C(\omega).
\end{aligned}$$

Hence,

$$\hat{u}_C(\omega,t) = \hat{f}_C(\omega) e^{-c^2\omega^2 t}.$$

The Fourier cosine transform of $f(x)$ is

$$\hat{f}_C(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(p)\cos(\omega p)\,\mathrm{d}p.$$

Hence,

$$\hat{u}_C(\omega,t) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(p)\cos(\omega p)\,\mathrm{d}p e^{-c^2\omega^2 t}.$$

To find the solution, all that is left is taking the inverse Fourier transform giving us:

$$u(x,t) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{u}_C(\omega,t)\cos(\omega x)\,\mathrm{d}\omega$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} \int_0^\infty f(p) \cos(\omega p) \, \mathrm{d}p \, e^{-c^2 \omega^2 t} \cos(\omega x) \, \mathrm{d}\omega$$
$$= \frac{2}{\pi} \int_0^\infty \int_0^\infty f(p) \cos(\omega p) e^{-c^2 \omega^2 t} \cos(\omega x) \, \mathrm{d}p \, \mathrm{d}\omega.$$

$$\boxed{u(x,t) = \frac{2}{\pi} \int_0^\infty \int_0^\infty f(p) \cos(\omega p) e^{-c^2 \omega^2 t} \cos(\omega x) \, \mathrm{d}p \, \mathrm{d}\omega}$$ RESULT

Lecture 11: Circular membrane, Fourier-Bessel series.

21. November 2025

Problem 11.1

Derive the expression

$$\overline{u}_x(r,\theta,\Phi) = \overline{u}_r(r,\theta,\Phi)\cos\theta\sin\Phi - \overline{u}_\theta(r,\theta,\Phi)\frac{\sin\theta}{r\sin\Phi} + \overline{u}_\Phi(r,\theta,\Phi)\frac{\cos\theta\cos\Phi}{r}$$

where (r,θ,Φ) are spherical coordinates as defined in Section 8.2.2.

Derivation

Spherical coordinates are defined in Section 8.2.2 as

$$r = \sqrt{x^2+y^2+z^2}, \qquad \cos\Phi = \frac{z}{\sqrt{x^2+y^2+z^2}}, \qquad \tan\theta = \frac{y}{x}.$$

From the Chain rule (p. 2) we have that

$$\overline{u}_x(r,\theta,\Phi) = \overline{u}_r(r,\theta,\Phi)r_x(x,y,z) + \overline{u}_\theta(r,\theta,\Phi)\theta_x(x,y,z) + \overline{u}_\Phi(r,\theta,\Phi)\Phi_x(x,y,z).$$

Therefore, we must find the derivative of each coordinate with respect to x as

$$\begin{aligned}
r_x(x,y,z) &= \frac{\partial\sqrt{x^2+y^2+z^2}}{\partial x} \\
\frac{\partial\sqrt{x^2+y^2+z^2}}{\partial x} &= \frac{x}{\sqrt{x^2+y^2+z^2}} = \cos\Phi\sin\theta \\
\frac{\partial\tan(\theta(x,y,z)}{\partial x} &= \frac{1}{\cos^2\theta(x,y,z)}\theta_x(x,y,z) \\
\frac{\partial\tan(\theta(x,y,z)}{\partial x} &= \frac{\partial}{\partial x}\frac{y}{x} \\
&= -\frac{y}{x^2} \\
-\frac{y}{x^2} &= -\frac{\sin\theta}{r\cos^2\theta\sin\Phi} \\
\implies \theta_x(x,y,z) &= -\frac{\sin\theta}{r\sin\Phi} \\
\frac{\partial\cos\Phi(x,y,z)}{\partial x} &= -\sin\Phi(x,y,z)\Phi_x(x,y,z) \\
\frac{\partial\cos\Phi(x,y,z)}{\partial x} &= \frac{\partial}{\partial x}\frac{z}{\sqrt{x^2+y^2+z^2}} \\
&= -\frac{xz}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}} \\
&= -\frac{\cos\theta\sin\Phi\cos\Phi}{r} \\
\implies \Phi_x(x,y,z) &= \frac{\cos\theta\cos\phi}{r}.
\end{aligned}$$

By insertion we get:

$$\overline{u}_x(r,\theta,\Phi) = \overline{u}(r,\theta,\Phi)\cos\Phi\sin\theta - \overline{u}_\theta(r,\theta,\Phi)\frac{\sin\theta}{r\sin\Phi} + \overline{u}_\Phi(r,\theta,\Phi)\frac{\cos\theta\cos\Phi}{r}.$$

Hence, it is shown.

Problem 11.2

Derive the expression

$$\overline{u}_y(r,\theta,\Phi) = \overline{u}_r(r,\theta,\Phi)\sin\theta\sin\Phi + \overline{u}_\theta(r,\theta,\Phi)\frac{\cos\theta}{r\sin\Phi} + \overline{u}_\Phi(r,\theta,\Phi)\frac{\sin\theta\cos\Phi}{r}$$

where (r,θ,Φ) are spherical coordinates as defined in Section 8.2.2.

Derivation

Following the same logic as in Problem 11.1 we take the derivatives in y this time as

$$\begin{aligned}
\frac{\partial\sqrt{x^2+y^2+z^2}}{\partial y} &= \frac{y}{\sqrt{x^2+y^2+z^2}} = \sin\theta\sin\Phi \\
\frac{\partial\tan\theta(x,y,z)}{\partial y} &= \frac{1}{x} = \frac{1}{r\cos\theta\sin\Phi} \\
\implies \theta_y(x,y,z) &= \frac{\cos\theta}{r\sin\Phi} \\
\frac{\partial\cos\Phi(x,y,z)}{\partial y} &= -\frac{yz}{\sqrt{x^2+y^2+z^2}^{\frac{3}{2}}} = -\frac{\sin\theta\sin\Phi\cos\Phi}{r} \\
\implies \Phi_y(x,y,z) &= \frac{\sin\theta\cos\Phi}{r}.
\end{aligned}$$

Inserting this, we get

$$\overline{u}_y(r,\theta,\Phi) = \overline{u}_r(r,\theta,\Phi)\sin\theta\sin\Phi + \overline{u}_\theta(r,\theta,\Phi)\frac{\cos\theta}{r\sin\Phi} + \overline{u}_\Phi(r,\theta,\Phi)\frac{\sin\theta\cos\Phi}{r}.$$

Hence, it is shown.

Problem 11.3

Show that Equations (1) and (2) are equivalent.

Derivation

Using the Laplacian, we rewrite (2), as

$$\begin{aligned}
\nabla^2 &= \frac{1}{r^2}\left(\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{\sin\Phi}\frac{\partial}{\partial\Phi}\left(\sin(\Phi)\frac{\partial}{\partial\Phi}\right) + \frac{1}{\sin^2\Phi}\frac{\partial^2}{\partial\theta^2}\right) \\
&= \frac{1}{r^2}\left(2r\frac{\partial}{\partial r} + r^2\frac{\partial^2}{\partial r^2} + \frac{1}{\sin\Phi}\left(\cos(\Phi)\frac{\partial}{\partial\Phi} + \sin(\Phi)\frac{\partial^2}{\partial\Phi^2}\right) + \frac{1}{\sin^2\Phi}\frac{\partial^2}{\partial\theta^2}\right) \\
&= \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\Phi^2} + \frac{1}{r^2}\cot(\Phi)\frac{\partial}{\partial\Phi} + \frac{1}{r^2\sin^2\Phi}\frac{\partial^2}{\partial\theta^2}.
\end{aligned}$$

Which is equivalent to the Laplacian of (1).

Problem 11.4

How does the general solution in Section 9.4 change if we are dealing with the two-dimensional heat equation on a disk of radius R subject to the condition that the boundary curve is kept at zero temperature and the initial temperature profile only depends on the distance from the centre of the disk.

Derivation

From the Lecture (p. 5) the general solution is

$$\overline{u}(r,t) = \sum_{m=1}^{\infty} \left(A_m \cos(\lambda_m t) + B_m \sin(\lambda_m t)\right) J_0\left(\frac{\alpha_m}{R} r\right).$$

In this case, our ODE for $G(t)$ is changed to

$$G_t(t) + \lambda_m^2 G(t) = 0, \quad \lambda_m = ck_m = c\frac{\alpha_m}{R}.$$

Which has solutions

$$G_m(t) = A_m e^{-\lambda_m^2 t}, \quad \lambda_m = ck_m = c\frac{\alpha_m}{R}.$$

Hence, the general solution is:

$$\boxed{\overline{u}(r,t) = \sum_{m=1}^{\infty} A_m e^{-\lambda_m^2 t} J_0\left(\frac{\alpha_m}{R} r\right)} \qquad \text{RESULT}$$

Lecture 12: Numerics for PDEs

28. November 2025

Problem 12.1

Find an approximation for $u_{xy}(x, y)$.

Derivation

We have that $u_{xy}(x, y) = (u_y(x, y))_x$. Using the approximation of partial derivatives from p. 3, we know that

$$\begin{aligned} u_x(x,y) &\approx \frac{u(x+h,y)-u((x-h),y)}{2h} \\ \big(u_x(x,y)\big)_y &\approx \frac{u_y(x+h,y)-u_y(x-h,y)}{2h} \\ u_{xy}(x,y) &\approx \frac{\frac{u(x+h,y+h)-u(x+h,y-h)}{2h}-\frac{u(x-h,y+h)-u(x-h,y-h)}{2h}}{2h} \\ &= \frac{u(x+h,y+h)-u(x+h,y-h)-u(x-h,y+h)-u(x-h,y-h)}{4h^2}. \end{aligned}$$

$$\boxed{u_{xy}(x,y) \approx \frac{u(x+h,y+h)-u(x+h,y-h)-u(x-h,y+h)-u(x-h,y-h)}{4h^2}}$$

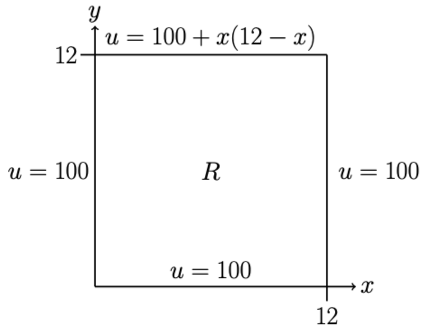
RESULT

Problem 12.2

Consider the Poisson equation

$$u_{xx}(x,y) + u_{yy}(x,y) = \frac{xy(12-x)(12-y)}{16}$$

and the situation depicted on the figure. Find the corresponding system of difference equations for mesh size $h = 4\,\text{cm}$.



Derivation

From the Lecture (p. 3) the corresponding difference equation in mesh notation is

$$u_{i+1,j} + u_{i,j+1} + u_{i-1,j} + u_{i,j-1} - 4u_{i,j} = \frac{ijh^4(12-ih)(12-jh)}{16}.$$

Our boundary mesh points are

$$\begin{aligned}
&u_{00} = 100, && u_{10} = 100, && u_{20} = 100, && u_{30} = 100,\\
&u_{01} = 100, && u_{31} = 100, && u_{02} = 100, && u_{32} = 100,\\
&u_{03} = 100, && u_{31} = 132, && u_{32} = 132, && u_{33} = 100.
\end{aligned}$$

We remember that our difference equation for our unknown points are

$$u_{i+1,j} + u_{i,j+1} + u_{i-1,j} + u_{i,j-1} - 4u_{i,j} = \frac{ijh^4(12-ih)(12-jh)}{16}.$$

Using this we start by finding the inner meshpoint u_{11} as

$$\begin{aligned}
u_{2,1} + u_{1,2} + u_{0,1} + u_{1,0} - 4u_{1,1} &= \frac{1 \cdot 1 \cdot 4^4(12-4)(12-4)}{16}\\
u_{2,1} + u_{1,2} + 100 + 100 - 4u_{1,1} &= 1024\\
-4u_{1,1} + u_{2,1} + u_{1,2} &= 824.
\end{aligned}$$

Now we find the meshpoint u_{21} as

$$\begin{aligned}
u_{3,1} + u_{2,2} + u_{1,1} + u_{2,0} - 4u_{2,1} &= \frac{2 \cdot 1 \cdot 4^4(12-8)(12-4)}{16}\\
100 + u_{2,2} + u_{1,1} + 100 - 4u(2,1) &= 1024\\
-4u_{2,1} + u_{2,2} + u_{1,1} &= 824.
\end{aligned}$$

Now we find the meshpoint u_{12} as

$$\begin{aligned}
u_{2,2} + u_{1,3} + u_{0,2} + u_{1,1} - 4u_{1,2} &= \frac{1 \cdot 2 \cdot 4^4(12-4)(12-8)}{16}\\
u_{2,2} + 132 + 100 + u_{1,1} - 4u_{1,2} &= 1024\\
-4u_{1,2} + u_{2,2} + u_{1,1} &= 792.
\end{aligned}$$

Lastly, we find the meshpoint u_{22} as

$$\begin{aligned} u_{3,2} + u_{2,3} + u_{1,2} + u_{2,1} - 4u_{2,2} &= \frac{2 \cdot 2 \cdot 4^4 (12-8)(12-8)}{16} \\ 100 + 132 + u_{1,2} + u_{2,1} - 4u_{2,2} &= 1024 \\ -4u_{2,2} + u_{1,2} + u_{2,1} &= 792. \end{aligned}$$

Hence,

$$\boxed{\begin{aligned} -4u_{1,1} + u_{2,1} + u_{1,2} &= 824 \\ -4u_{2,1} + u_{2,2} + u_{1,1} &= 824 \\ -4u_{1,2} + u_{2,2} + u_{1,1} &= 792 \\ -4u_{2,2} + u_{1,2} + u_{2,1} &= 792. \end{aligned}}$$

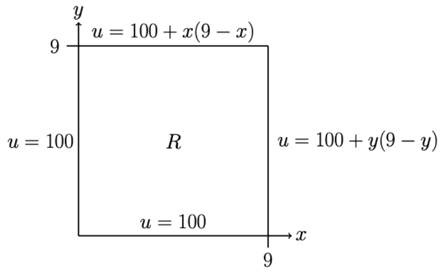
RESULT

Problem 12.3

Consider the Laplace equation

$$u_{xx}(x,y) + u_{yy}(x,y) = 0$$

and the situation as depicted on the figure. Find the corresponding system of difference equations for mesh size $h = 3$.



Derivation

The corresponding difference equation is (p. 5)

$$u_{i+1,j} + u_{i,j+1} + u_{i-1,j} + u_{i,j-1} - 4u_{i,j} = 0.$$

Here, our boundary meshpoints are

$$\begin{aligned} u_{00} &= 100, & u_{10} &= 100, & u_{20} &= 100, & u_{30} &= 100, \\ u_{01} &= 100, & u_{31} &= 118, & u_{02} &= 100, & u_{32} &= 118, \\ u_{03} &= 100, & u_{13} &= 118, & u_{23} &= 118, & u_{33} &= 100. \end{aligned}$$

We now find meshpoint u_{11} as

$$\begin{aligned} u_{21} + u_{12} + u_{01} + u_{10} - 4u_{11} &= 0 \\ u_{21} + u_{12} + 100 + 100 - 4u_{11} &= 0 \\ -4u_{11} + u_{12} + u_{21} &= -200. \end{aligned}$$

Moving on to meshpoint u_{21} we get

$$\begin{aligned} u_{31} + u_{22} + u_{11} + u_{20} - 4u_{21} &= 0 \\ 118 + u_{22} + u_{11} + 100 - 4u_{21} &= 0 \\ u_{11} - 4u_{21} + u_{22} &= -218. \end{aligned}$$

For meshpoint u_{12} we get

$$\begin{aligned} u_{22} + u_{13} + u_{02} + u_{11} - 4u_{12} &= 0 \\ u_{22} + 118 + 100 + u_{11} - 4u_{12} &= 0 \\ u_{11} - 4u_{12} + u_{22} &= -218. \end{aligned}$$

And for meshpoint u_{22} we get

$$\begin{aligned} u_{32} + u_{23} + u_{12} + u_{21} - 4u_{22} &= 0 \\ 118 + 118 + u_{12} + u_{21} - 4u_{22} &= 0 \\ u_{12} + u_{21} - 4u_{22} &= -236. \end{aligned}$$

$$\boxed{\begin{aligned} -4u_{11} + u_{12} + u_{21} &= -200 \\ u_{11} - 4u_{21} + u_{22} &= -218 \\ u_{11} - 4u_{12} + u_{22} &= -218 \\ u_{12} + u_{21} - 4u_{22} &= -236. \end{aligned}}$$

RESULT

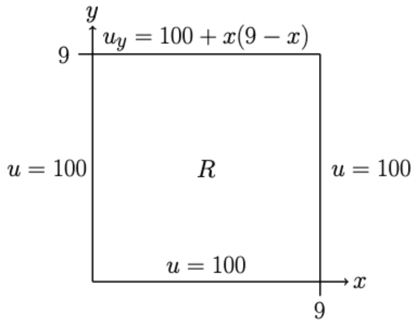
Lecture 13: Neumann and Mixed Boundary Conditions. Irregular Boundary. 5. December 2025

Problem 13.1

Consider the two-dimensional Laplace equation

$$u_{xx}(x,y) + u_{yy}(x,y) = 0$$

on the region R with mixed boundary conditions as depicted on the figure. Find the corresponding system of approximate difference equations for mesh size $h = 3$.



Derivation

The corresponding difference equation is (L. 12 p. 5):

$$u_{i+1,j} + u_{i,j+1} + u_{i-1,j} + u_{i,j-1} - 4u_{i,j} = 0.$$

Our boundary meshpoints are

$$\begin{aligned} &u_{00} = 100, && u_{10} = 100, && u_{20} = 100, && u_{30} = 100, \\ &u_{01} = 100, && u_{31} = 100, && u_{02} = 100, && u_{32} = 100, \\ & && u_{03} = 100, && u_{33} = 100. && \end{aligned}$$

The inner meshpoint P_{11} is found as

$$\begin{aligned} u_{21} + u_{12} + u_{01} + u_{10} - 4u_{11} &= 0 \\ u_{21} + u_{12} + 100 + 100 - 4u_{11} &= 0 \\ -4u_{11} + u_{21} + u_{12} &= -200. \end{aligned}$$

Now, the inner meshpoint P_{21} is found as

$$\begin{aligned} u_{31} + u_{22} + u_{11} + u_{20} - 4u_{21} &= 0 \\ 100 + u_{22} + u_{11} + 100 - 4u_{21} &= 0 \\ u_{11} - 4u_{21} + u_{22} &= -200. \end{aligned}$$

Now P_{12} is found as

$$\begin{aligned} u_{22} + u_{13} + u_{02} + u_{11} - 4u_{12} &= 0 \\ u_{22} + u_{31} + 100 + 100 - 4u_{12} &= 0 \\ u_{11} - 4u_{12} + u_{22} + u_{31} &= -200. \end{aligned}$$

Now we find P_{22} as

$$\begin{aligned} u_{32} + u_{23} + u_{12} + u_{21} - 4u_{22} &= 0 \\ 100 + u_{23} + u_{12} + u_{21} - 4u_{22} &= 0 \\ u_{21} + u_{12} - 4u_{22} + u_{23} &= -100. \end{aligned}$$

Now we move on to the boundary point P_{13}, which we find as

$$\begin{aligned} u_{23} + u_{14} + u_{03} + u_{12} - 4u_{13} &= 0 \\ u_{23} + u_{14} + 100 + u_{12} - 4u_{13} &= 0 \\ u_{12} - 4u_{13} + u_{23} + u_{14} &= -100. \end{aligned}$$

To find an expression for u_{14}, we use the directional derivative as

$$\begin{aligned} 118 = u_n(P_{13}) = \frac{\partial u_{13}}{\partial y} \approx \frac{u_{14} - u_{12}}{2h} &= \frac{u_{14} - u_{12}}{6} \\ \implies u_{14} &= u_{12} + 708. \end{aligned}$$

We insert this into the previous expression to get

$$\begin{aligned} u_{12} - 4u_{13} + u_{23} + u_{12} + 708 &= -100 \\ 2u_{12} - 4u_{13} + u_{23} &= -808. \end{aligned}$$

Lastly we find the boundary point P_{23} as

$$\begin{aligned} u_{33} + u_{24} + u_{13} + u_{22} - 4u_{23} &= 0 \\ 100 + u_{24} + u_{13} + u_{22} - 4u_{23} &= 0 \\ u_{13} + u_{22} - 4u_{23} + u_{24} &= -100. \end{aligned}$$

We, once again, use the directional derivative as

$$\begin{aligned} 118 = u_n(P_{23}) = \frac{\partial u_{23}}{\partial y} \approx \frac{u_{24} - u_{22}}{2h} &= \frac{u_{24} - u_{22}}{6} \\ u_{24} &= u_{22} + 708. \end{aligned}$$

And by insertion we get

$$\begin{aligned} u_{13} + u_{22} - 4u_{23} + u_{22} + 708 &= -100 \\ 2u_{22} + u_{13} - 4u_{23} &= -808. \end{aligned}$$

$$\boxed{\begin{aligned} -4u_{11} + u_{21} + u_{12} &= -200 \\ u_{11} - 4u_{21} + u_{22} &= -200 \\ u_{11} - 4u_{12} + u_{22} + u_{31} &= -100 \\ u_{21} + u_{12} - 4u_{22} + u_{23} &= -100 \\ 2u_{12} - 4u_{13} + u_{23} &= -808 \\ 2u_{22} + u_{13} - 4u_{23} &= -808. \end{aligned}}$$

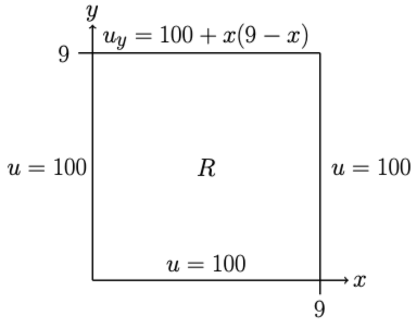
RESULT

Problem 13.2

Consider the two-dimensional Poisson equation

$$u_{xx}(x,y) + u_{yy}(x,y) = x + y$$

on the region R with mixed boundary conditions as depicted on the figure. Find the corresponding system of approximate solutions for mesh size $h = 4{,}5$.



Derivation

We have the difference equation in mesh notation:

$$u_{i+1,j} + u_{i,j+1} + u_{i-1,j} + u_{i,j-1} - 4u_{ij} = h^2(ih + jh).$$

The known boundary mesh points are

$$u_{02} = 100, \qquad u_{10} = 100, \qquad u_{20} = 100,$$
$$u_{01} = 100, \qquad u_{21} = 100, \qquad u_{22} = 100.$$

Now, we find the inner meshpoint P_{11} as

$$\begin{aligned} u_{21} + u_{12} + u_{01} + u_{10} - 4u_{11} &= 4{,}5^2 \cdot (1 \cdot 4{,}5 + 1 \cdot 4{,}5) \\ 100 + u_{12} + 100 + 100 - 4u_{11} &= 182{,}25 \\ -4u_{11} + u_{12} &= -117{,}75. \end{aligned}$$

The boundary point P_{12} is found as

$$\begin{aligned} u_{22} + u_{13} + u_{02} + u_{11} - 4u_{12} &= 4{,}5^2 \cdot (1 \cdot 4{,}5 + 2 \cdot 4{,}5) \\ 100 + u_{13} + 100 + u_{11} - 4u_{12} &= 273{,}375 \\ u_{11} - 4u_{12} + u_{13} &= 73{,}375. \end{aligned}$$

To find an expression for u_{13} we use the directional derivative as:

$$\begin{aligned} 120{,}25 = u_n(P_{12}) = \frac{\partial u_{12}}{\partial y} &\approx \frac{u_{13} - u_{11}}{9} \\ u_{13} &= u_{11} + 1082{,}25. \end{aligned}$$

By insertion we get:

$$\begin{aligned} u_{11} - 4u_{12} + u_{11} + 1082{,}25 &= 73{,}375 \\ 2u_{11} - 4u_{12} &= -1008{,}875. \end{aligned}$$

$$\boxed{\begin{aligned} -4u_{11} + u_{12} &= -117{,}75 \\ 2u_{11} - 4u_{12} &= -1008{,}875. \end{aligned}}$$

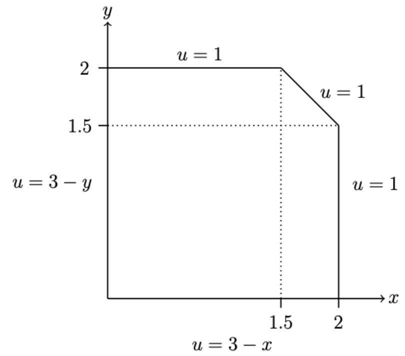
RESULT

Problem 13.3

Consider the two-dimensional Laplace equation

$$u_{xx}(x,y)+u_{yy}(x,y)=0$$

on the region R with boundary conditions as depicted on the figure. Find the approximate solution for mesh size $h=1{,}5$.



Derivation

We know the boundary values:

$$u_{10}=1{,}5,\qquad u_{01}=1{,}5,\qquad u(A)=1,\qquad u(B)=1.$$

We now approximate $\nabla^2 u(P_{11})$ using $a=b=0{,}5/1{,}5=1/3$ in the formula from the Lecture (p. 5):

$$\begin{aligned}\nabla^2 u(P_{11}) &\approx \frac{2}{1{,}5^2}\left(\frac{9}{4}u(A)+\frac{9}{4}u(B)+\frac{3}{4}u(P_{01})+\frac{3}{4}u(P_{10})-6u(P_{11})\right)\\ &= \frac{2}{1{,}5^2}\left(\frac{9}{4}+\frac{9}{4}+\frac{3}{4}\cdot 1{,}5+\frac{3}{4}\cdot 1{,}5-6u_{11}\right).\end{aligned}$$

Hence, our difference equation is

$$\begin{aligned}\nabla^2 u(P_{11}) &= 0\\ \frac{2}{1{,}5^2}\left(\frac{9}{4}+\frac{9}{4}+\frac{3}{4}\cdot 1{,}5+\frac{3}{4}\cdot 1{,}5-6u_{11}\right) &= 0\\ 6{,}75-6u_{11} &= 0\\ u_{11} &\approx 1{,}125.\end{aligned}$$

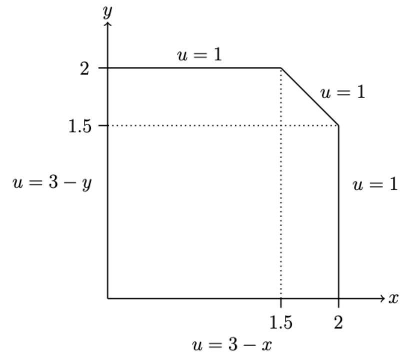
$$\boxed{u_{11}\approx 1{,}125}$$ Result

Problem 13.4

Consider the two-dimensional Poisson equation

$$u_{xx}(x,y) + u_{yy}(x,y) = x + y$$

on the region R with boundary conditions as depicted on the figure. Find the approximate solution for mesh size $h = 1{,}5$.



Derivation

In this case, we have the same boundary values:

$$u_{10} = 1{,}5, \qquad u_{01} = 1{,}5, \qquad u(A) = 1, \qquad u(B) = 1$$

and the same approximation of $\nabla^2 u(P_{11})$ as above:

$$\nabla^2 u(P_{11}) \approx \frac{2}{1{,}5^2}\left(\frac{9}{4} + \frac{9}{4} + \frac{3}{4}\cdot 1{,}5 + \frac{3}{4}\cdot 1{,}5 - 6u_{11}\right).$$

In this case, however, our difference equation becomes:

$$\begin{aligned} \nabla^2 u(P_{11}) &= h + h \\ 6{,}75 - 6u_{11} &= 3\cdot\frac{1{,}5^2}{2} \\ u_{11} &\approx 0{,}5625. \end{aligned}$$

$$\boxed{u_{11} \approx 0{,}5625}$$ RESULT