Statics and Strength of Materials

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Contents

Lecture 1: Intro + Force & position vectors 25. August 2025

1 Introduction

This is my personal notes for the Statics and strength of materials course taught at Aarhus University by Tito Andriollo and Souhayl Sadik.

1.1 Terminology

Some basic terminology must be defined before we can start.

Definition 1: Particle

A particle is a body whose geometry can be neglected in the problem at hand.

Definition 2: Rigid body

A rigid body is a combination of a large number of particles with a fixed distance from each other. I.e. immovable atoms.

Definition 3: Deformable body

A deformable body is a combination of a large number of particles, but where the distance between the particle can vary subject to forces.

2 Force vectors

2.1 Scalars and vectors

Oftentimes in engineering mechanics one utilizes scalars or vectors to measure physical quantities.

Definition 4: Scalar

A *scalar* is any positive or negative numerical quantity – i.e. any physical quantity that is expressable solely by its *magnitude*. Scalar quantities include length, mass, time, etc.

Definition 5: Vector

A *vector* is any physical quantity that requires both a *magnitude* and a *direction* to be fully described. Vector quantities include force, moment, velocity, etc. In this collection of notes vector quantities will be represented by boldface notation such as $\mathbf{A}$.

2.2 Vector operations

2.2.1 Multiplication and division of a vector by a scalar

When one multiplies or divides a vector quantity by a scalar its magnitude is simply changed by that amount. The direction will remain the same.

2.2.2 Vector addition and subtraction

When adding two vector quantities one must both account for the magnitudes and the directions of the vector quantities. To do this graphically one places the tail end one of one of the vectors at the head end of the other – the endpoint is their sum. Algebraically this corresponds to adding the component vectors pairwise.

For subtraction one can once again use the algebraic method of the pairwise components or graphically one can place the tail ends of the vectors at the same point and find the 'difference vector' that combines their heads.

2.3 Addition of a system of coplanar forces

When a force is resolved into components along the x and y axes, the components are called *rectangular components*. We can represent these either by *scalar notation* or *Cartesian notation*. Only cartesian notation will be covered in these notes.

2.3.1 Cartesian notation

One can represent a force $\mathbf{F}$ as a sum of the magnitudes of the force F_x and F_y and the cartesian unit vectors $\mathbf{i}$ and $\mathbf{j}$. That is:

$$\mathbf{F} = F_x\mathbf{i} + F_y\mathbf{j}.$$

2.3.2 Coplanar force resultants

Given three forces:

$$\begin{aligned}\mathbf{F}_1 &= F_{1x}\mathbf{i} + F_{1y}\mathbf{j}\\ \mathbf{F}_2 &= -F_{2x}\mathbf{i} + F_{2y}\mathbf{j}\\ \mathbf{F}_3 &= F_{3x}\mathbf{i} - F_{3y}\mathbf{j}.\end{aligned}$$

One can compute the vector resultant as described in subsubsection 2.2.2 as:

$$\begin{aligned}\mathbf{F}_R &= \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3\\ &= (F_{1x} - F_{2x} + F_{3x})\,\mathbf{i} + (F_{1y} + F_{2y} - F_{3y})\,\mathbf{j}\\ &= (F_r)_x\,\mathbf{i} + (F_r)_y\,\mathbf{j}.\end{aligned}$$

The magnitude of the resultant force can be found from the magnitudes of the resultant component vectors and the Pythagorean theorem as:

$$F_r = \sqrt{(F_r)_x^2 + (F_r)_y^2}.$$

The angle which specifies the direction of the resultant force can be determined using trigonometry as:

$$\theta = \tan^{-1}\left|\frac{(F_R)_y}{(F_R)_x}\right|.$$

The same principle applies in three dimensions.

2.4 Position vectors

A position vector $\mathbf{r}$ is defines as a fixed vector which locates a specific point in space relative to another specific point in space. A position vector between two forces with the same tail end point will correspond do the subtracyion of the vectors.

2.4.1 Force vector directed along a line

Oftentimes in three-dimensional statics problems, the direction of a force $\mathbf{F}$ is specified by two points through which its line of action passes. If we call these two points A and B we can formulate $\mathbf{F}$ by realizing that it has the same direction as the position vector $\mathbf{r}$ from point A to point B. The common direction is specified by the unit vector $\mathbf{u} = \mathbf{r}/r$. We get:

$$\mathbf{F} = F\mathbf{u} = F\left(\frac{\mathbf{r}}{r}\right) = F\left(\frac{(x_B - x_A)\,\mathbf{i} + (y_B - y_A)\,\mathbf{j} + (z_B - z_A)\,\mathbf{k}}{\sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2}}\right).$$

Lecture 2: Moment of a force

28. August 2025

2.5 Dot product

The dot product of vectors $\mathbf{A}$ and $\mathbf{B}$ is defined as:

$$\mathbf{A} \cdot \mathbf{B} = AB\cos\theta.$$

This is also called the scalar product of the vectors. The following laws of operation apply:

$$\begin{aligned} \mathbf{A} \cdot \mathbf{B} &= \mathbf{B} \cdot \mathbf{A} \\ a\,(\mathbf{A} \cdot \mathbf{B}) &= (a\mathbf{A}) \cdot \mathbf{B} = \mathbf{A} \cdot (a\mathbf{B}) \\ \mathbf{A} \cdot (\mathbf{B} + \mathbf{D}) &= (\mathbf{A} \cdot \mathbf{B}) + (\mathbf{A} \cdot \mathbf{D})\,. \end{aligned}$$

In Cartesian form the dot product can be expressed as:

$$\mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y + A_z B_z.$$

3 Force system resultants

3.1 Moment of a force – Scalar formulation

A force applied to a body will tend to produce a rotation about a point that is not on the line of action of the force – this phenomenon is called the *moment* or the *torque*.

In general, if we consider the force $\mathbf{F}$ and point O to lie in the shaded plane shown on **Figure 3.1**, then the moment $\mathbf{M}_O$ about the point O, or about an axis through O and perpendicular to the plane is a vector quantity since it has both a magnitude and a direction.

The magnitude of $\mathbf{M}_O$ is

$$M_O = Fd$$

where d is the *moment arm* or the perpendicular distance from the axis through point O to the line of action of the force.

The direction of $\mathbf{M}_O$ can be determined using the right hand rule – curl the fingers in the direction of rotation and the thumb will be pointing in the direction of the moment.

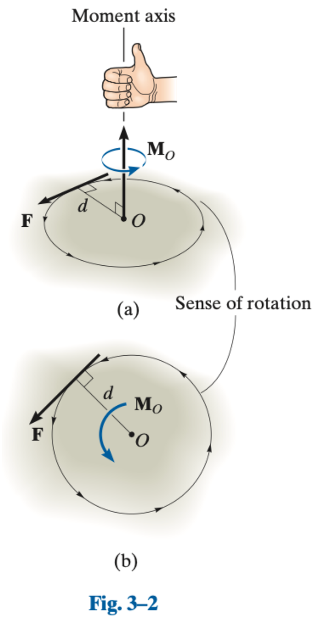

Figure 3.1:

The resultant moment of multiple forces lying in the same plane is simply found by their algebraic sum. I.e.

$$M_{R_O} = \sum Fd.$$

3.2 Principle of moments

A very useful concept often used in mechanics is the principle of moments, which is also known as Varignon's theorem.

Definition 6: Principle of moments

The moment of a force about a point is equal to the sum of the moments of the components of the force about the point.

This can be applied to the case shown on **Figure 3.2**. Here a force **F** is split into components and the resultant moment can then simply be found as:

$$M_O = F_x y + F_y x.$$

3.3 Cross product

The cross product of the two vectors **A** and **B** yields a vector **C** as:

$$\mathbf{C} = \mathbf{A} \times \mathbf{B}.$$

The magnitude of **C** is given by:

$$C = AB \sin\theta.$$

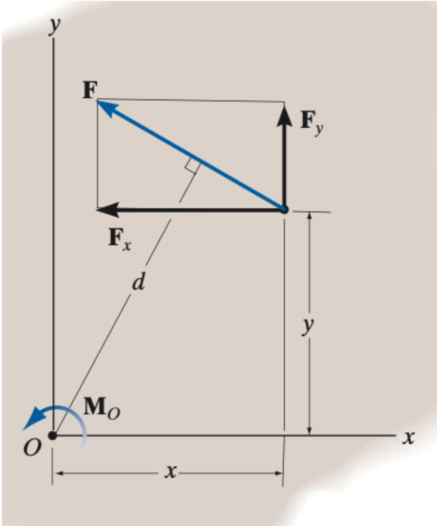

Figure 3.2:

The vector **C** will have a direction perpendicular to the plane containing **A** and **B** according to the right hand rule.

The following laws of operation apply:

$$\begin{aligned}\mathbf{A}\times\mathbf{B} &= -\mathbf{B}\times\mathbf{A}\\ a\left(\mathbf{A}\times\mathbf{B}\right) &= \left(a\mathbf{A}\right)\times\mathbf{B} = \mathbf{A}\times\left(a\mathbf{B}\right)\\ \mathbf{A}\times\left(\mathbf{B}+\mathbf{D}\right) &= \left(\mathbf{A}\times\mathbf{B}\right)+\left(\mathbf{A}\times\mathbf{D}\right).\end{aligned}$$

In Cartesian coordinates the cross product may be written as:

$$\mathbf{A}\times\mathbf{B} = \left(A_yB_z - A_zB_y\right)\mathbf{i} - \left(A_xB_z - A_zB_x\right)\mathbf{j} + \left(A_xB_y - A_yB_x\right)\mathbf{k}.$$

In determinant form it is:

$$\mathbf{A}\times\mathbf{B} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ A_x & A_y & A_z\\ B_x & B_y & B_z\end{vmatrix}.$$

3.4 Moment of a force – Vector formulation

The moment of a force **F** about point O can be expressed using the vector cross product as:

$$\mathbf{M}_O = \mathbf{r}\times\mathbf{F}.$$

Here **r** is a position vector *from O to any point* on the line of action of **F**. When using this formulation $\mathbf{M}_O$ will have the correct magnitude and direction no matter what position vector is chosen along as the previous rule is respected.

The magnitude of this is:

$$M_O = rF\sin\theta = F\left(r\sin\theta\right) = Fd$$

and the direction of $\mathbf{M}_O$ is once again determined by the right hand rule.

The cross product is typically used in three dimensions since the perpendicular distance or moment arm from point O to the line of action of the force is not needed. In other words any position vector $\mathbf{r}_i$ pointing from the point O to the line of action of $\mathbf{F}$ is adequate.

If a body is acted upon by a system of forces, the resultant moment of the forces about point O can be determined by vector addition of the moment of each force as:

$$\mathbf{M}_{R_O} = \sum (\mathbf{r} \times \mathbf{F}) .$$

Lecture 3: Systems of forces and moments 1. September 2025

4 Force System Resultants

4.1 Moment of a couple

A *couple* is defined as two parallel forces that hav equal magnitude but opposite directions and are separated by a distance d as shown on **Figure 4.1**. Since the resultant force of the couple is zero the only effect is to produce a tendency for rotation.

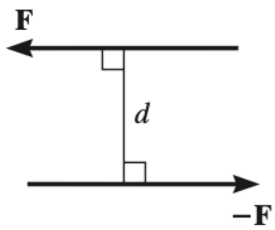

Fig. 3–25

Figure 4.1:

The moment produced by a couple is called a *couple moment* it can be shown that this is equal to:

$$\mathbf{M} = \mathbf{r} \cdot \mathbf{F}.$$

This is a *free vector*, i.e. it can act at *any point* since $\mathbf{M}$ only depends on the position vector r between the forces and not the position vectors from the origin.

4.2 Simplification of a force and couple system

It is often convenient to reduce a complex system of forces and couple moments acting on a body to a simpler form by replacing it with an equivalent system. A system is equivalent if the *external effects* it produces on a body are the same as those caused by the original force and moment system.

Every system of several forces and couple moments can be broken down into a single resultant force acting at a point O and a resultant couple moment. This can be done by adding together the forces in each Cartesian direction as well as the moments each on their own.

4.3 Reduction of a simple distributed loading

Sometimes, a body is subjected to a load distributed over its surface. The pressure caused by this distributed loading on the surface represents the loading intensity and is measured in Pa.

The most common type of distributed loading is that along a single axis. Take for example a beam of constant width b subjected to a pressure loading that varies along the x-axis. The loading is described by the function $p = p(x)$. Since this contains only one variable we can quickly "one-dimensionalize" the problem by $w(x) = p(x)b$.

The magnitude of the resultant force in this case must be found using integration as summing "all the forces" would require an infinte sum. I.e.

$$F_R = \int_L w(x)\,\mathrm{d}x = \int_A \mathrm{d}A = A.$$

The location $\overline{x}$ of $\mathbf{F}_R$ can be determined using static equilibrium and moments as:

$$-\overline{x}F_R = -\int_L xw(x)\,\mathrm{d}x.$$

And solving for $\overline{x}$ we have:

$$\overline{x} = \frac{\int_L xw(x)\,\mathrm{d}x}{\int_L w(x)\,\mathrm{d}x} = \frac{\int_A x\,\mathrm{d}A}{\int_A \mathrm{d}A}.$$

Lecture 4: Center of Gravity 4. September 2025

5 Center of Gravity and Centroid

5.1 Center of Gravity and the centroid of a body

A body can be thought of as consisting of an infinite amount of differential particles (actually bodies consist of discrete atoms, but it is helpful to think of it in the infinite sense). If a body is located in a gravitational field, each of the particles will be subject to a weight $\mathrm{d}W$. These weights will form an approximately parallel system of forces and the resultant of this system is the total weight of the body, which passes through a single point called the center of gravity.

We consider the rod on Figure 5.1, where the segment with weight $\mathrm{d}W$ is located at an arbitrary position $\tilde{x}$. The total weight of the rod is then:

$$W = \int \mathrm{d}W.$$

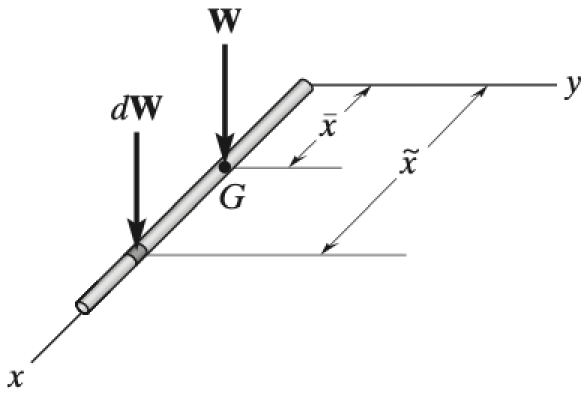

Figure 5.1:

The location of the canter of gravity, measured from the y-axis is determined by equating the moment of W about the y-axis to the sum of the moments of the weights of all its particles about said axis. I.e.

$$\overline{x}W = \int \overline{x}\,\mathrm{d}W \implies \overline{x} = \frac{\int \overline{x}\,\mathrm{d}W}{\mathrm{d}W}.$$

This same procedure can be applied for all axes, i.e.

$$\overline{x} = \frac{\int \overline{x}\,\mathrm{d}W}{\int \mathrm{d}W}, \quad \overline{y} = \frac{\int \overline{y}\,\mathrm{d}W}{\int \mathrm{d}W}, \quad \overline{z} = \frac{\int \overline{z}\,\mathrm{d}W}{\int \mathrm{d}W}.$$

5.1.1 Centroid of a Volume

We consider the body on Figure 5.2. If it is made from a homogeneous material, then its specific weight γ will be constant. Therefore, a differential element of volume $\mathrm{d}V$ has a weight $\mathrm{d}W = \gamma\,\mathrm{d}V$. Substituting this into the equations from before, and cancelling out ρ we obtain the coordinates of the centroid C as:

$$\overline{x} = \frac{\int_V \tilde{x}\,\mathrm{d}V}{\int_V \mathrm{d}V}, \quad \overline{y} = \frac{\int_V \tilde{y}\,\mathrm{d}V}{\int_V \mathrm{d}V}, \quad \overline{z} = \frac{\int_V \tilde{z}\,\mathrm{d}V}{\int_V \mathrm{d}V}.$$

5.1.2 Centroid of an area

Similarly, if an area lies in the x-y-plane and is bounded by the curve $y = f(x)$ then its centroid will be in the same plane and can be determined as:

$$\overline{x} = \frac{\int_A \tilde{x}\,\mathrm{d}A}{\int_A \mathrm{d}A}, \quad \overline{y} = \frac{\int_A \tilde{y}\,\mathrm{d}A}{\int_A \mathrm{d}A}.$$

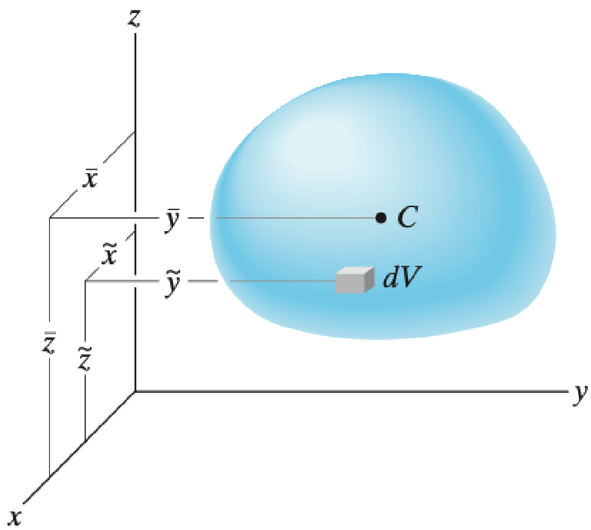

Figure 5.2:

5.2 Composite Bodies

A composite body consists of a series of connected "simpler" bodies. Provided the weight and location of center of gravity of each part we can eliminate the need for integration as:

$$\overline{x} = \frac{\sum \tilde{x} W}{\sum W}, \quad \overline{y} = \frac{\sum \tilde{y} W}{\sum W}, \quad \overline{z} = \frac{\sum \tilde{z} W}{\sum W}.$$

Lecture 5: Rigid Body Equilibrium, Pt. 1 8. September 2025

6 Equilibrium of a Rigid Body

6.1 Conditions for Rigid-Body Equilibrium

As mentioned previously the force and couple moments acting on a body can be reduced to an equivalent resultant force and resultant couple moment at any arbitrary point O on or off the body. If these resultants are both equal to zero the body is said to be in equilibrium. Mathematically this is:

$$\mathbf{F}_R = \sum \mathbf{F} = \mathbf{0}$$
$$(\mathbf{M}_R)_= = \sum \mathbf{M}_O = \mathbf{0}.$$

These two equations are both necessary and sufficient for equilibrium. This is true in both 2 and 3 dimensions.

6.2 Characteristics of Dry Friction

Friction is the name of the force that resists the movement between two contacting surfaces that slide relative to each other. Here only dry friction, also called *Coulomb friction*, is described. This occurs when there is no lubricating liquid between the two contacting surfaces.

This force will resist motion caused by an applied force **P** when the size of the applied force **P** increases above the *limiting static frictional force*, F_s, motion will occur. F_s is given by:

$$F_s = \mu_s N$$

where μ_s is the coefficient of static friction and N is the normal force. When the applied force **P** increases above F_s and motion occurs the friction force will drop to a smaller value called the *kinetic frictional force* given by:

$$F_k = \mu_k N$$

where μ_k is the coefficient of kinetic friction.

Lecture 6: Free-body diagrams and equilibrium equations 3D

11. September 2025

6.3 Equations of equilibrium

Above it is noted that the necessary and sufficient conditions for equilibrium for a rigid body is $\sum \mathbf{F} = 0$ and $\sum \mathbf{M}_O = 0$. In two dimensions this is often written as:

$$\sum F_x = 0$$
$$\sum F_y = 0$$
$$\sum M_0 = 0.$$

Although the above representation is the most frequently used one, two alternate sets of three independent equilibrium equations may also be used. One of these is:

$$\sum F_x = 0$$
$$\sum M_A = 0$$
$$\sum M_B = 0.$$

Here M_A and M_B is the moments about an arbitrary point A and B, respectively. When applying these it is a requirement that a line passing through A and B is not parallel to the y-axis.

Another set of equilibrium equations is:

$$\sum M_B = 0$$
$$\sum M_C = 0.$$

Here it is a requirement that points A, B, and C do not lie on the same line.

6.4 Two- and Three-force members

The solutions to some equilibrium problems can be simplified by recognizing members that are only subjected to two or three forces.

6.4.1 Two-Force Members

A two-force member has forces applied at only two points. An example of this is shown on Figure 6.1. To satisfy force equilibrium, $\mathbf{F}_A$ and $\mathbf{F}_B$ must be equal in magnitude but opposite in direction. Furthermore, moment equilibrium requires that $\mathbf{F}_A$ and $\mathbf{F}_B$ share the same line of action, which can only happen if they are directed along the line joining A and B.

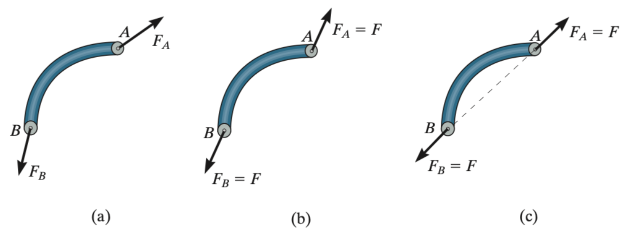

Figure 6.1: Two-force member

6.4.2 Three-Force Members

A three-force member has forces applied at three different points. Moment equilibrium can only be satisfied if the three forces form a *concurrent* or *parallel* force system. These two cases are shown on Figure 6.2.

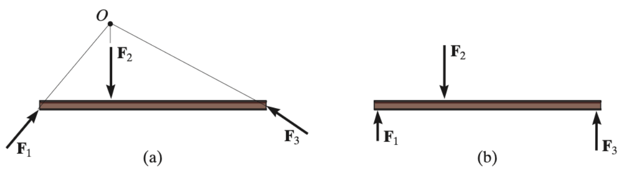

Figure 6.2: Three-force member

7 Equilibrium in Three Dimensions

7.1 Equations of Equilibrium

In three dimensions the vector equations of equilibrium may be stated as:

$$\sum \mathbf{F} = \mathbf{0}$$
$$\sum \mathbf{M}_O = \mathbf{0}.$$

In cartesian form this is:

$$\sum \mathbf{F} = \sum F_x \mathbf{i} + \sum F_y \mathbf{j} + \sum F_z \mathbf{k} = \mathbf{0}$$
$$\sum \mathbf{M}_O = \sum M_x \mathbf{i} + \sum M_y \mathbf{j} + \sum M_z \mathbf{k} = \mathbf{0}.$$

This can be broken down into six scalar equations, which may be used to solve for at most six unknowns.

Lecture 7: Structural Analysis – Simple trusses 15. September 2025

8 Structural Analysis

8.1 Simple Trusses

Definition 7: Truss

A *truss* is a structure consisting of slender members joined at their end points.
A special case of this is *planar trusses*, which lie in a single plane and are often used to support roofs and bridges.

When working with trusses it is assumed that all forces are applied at the joints, and that the members are joined by smooth pins – These two assumptions means that each truss member will act as a two-force member, i.e. either be in compression or tension.

8.1.1 Simple truss

If one connects three members at their ends they form a triangular truss as shown on Figure 8.1. This structure will always be rigid. Attaching the new connected members one can form a larger truss as shown on Figure 8.2. This procedure can be repeated and a *simple truss* will always be formed.

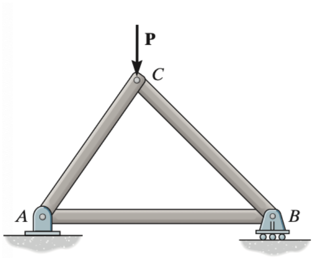

Figure 8.1: Triangular simple truss

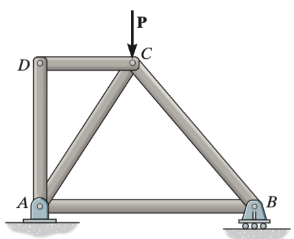

Figure 8.2: Extended version of the truss from Figure 8.1

8.2 The method of joints

One, commonly used, way to determine the force in each member of a truss is by using the method joints. This is based on the fact that if the entire truss is in equilibrium then each joint must also be in equilibrium. Therefore, one can use the force equilibrium equations in two dimensions on each joint.

When using the method joints, one must make sure to start at a joint having at least one known force and at most two unknown forces.

8.3 Zero-force members

Some members in a truss will under certain circumstances support *no* loading – these are termed *zero-force members*. In general, one can identify the zero-force members of a truss by inspection of the free-body diagrams for each joint. Also, it is important to remember that, if three members form a truss joint, for which two of the members are colinear, the third member is a zero-force member provided no external force or support reaction acts along this member.

Lecture 8: Structural Analysis – Indeterminate trusses 18. September 2025

8.4 The method of sections

When we are just interested in the force in a few members of the truss it is often found using the method of sections. This relies on the fact that for the truss to be in equilibrium then any part of the truss must also be in equilibrium. In this way, one can "cut" through a truss and analyze any part of it simply by applying the correct forces – keep in mind that since we only have three equations of equilibrium then the cut should not intersect more than three members in which the forces are unknown.

It is preferable to choose a point to sum the moments about which reduces the amount of forces the most possible.

Lecture 10: Internal Loading and Stress 25. September 2025

9 Stress and Strain

9.1 Internal Resultant Loadings

When one seeks to determine the internal loadings that act within a body a convenient method is to use the method of sections. E.g. for the body shown on Figure 9.1 it is necessary to pass an imaginary section through the region where the internal loadings are to be determined. Doing this the body is separated into two separate bodies each of which a free diagram can be constructed for.

It is convenient to split both the force and moment into components as shown on Figure 9.2. Here the force is split into a normal force component N, a shear force component V, a torque T and a bending moment M.

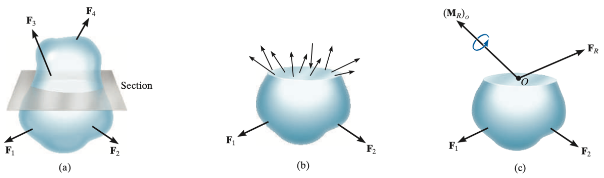

Figure 9.1: Body held in equilibrium by four external forces.

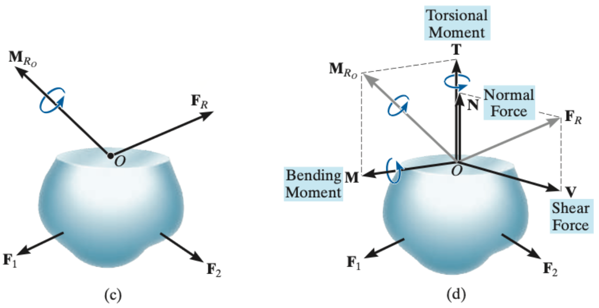

Figure 9.2: Components of force and moment

9.2 Stress

On fig. 9.2 the resultant internal force and moment acting at O represents the effects of the distributed loading over the sectioned area. To describe this distribution the concept of stress is often applied. As we know from materials science stress is defined as the limit

$$\sigma = \lim_{\Delta A \to 0} \frac{\Delta F}{\Delta A}.$$

Using this we can define a stress component in each of the coordinate directions (assuming $+z$ is upwards), namely a normal stress,

$$\sigma_z = \lim_{\Delta A \to 0} \frac{\Delta F_z}{\Delta A},$$

and two shear stresses

$$\tau_{zx} = \lim_{\Delta A \to 0} \frac{\Delta F_x}{\Delta A}$$
$$\tau_{zy} = \lim_{\Delta A \to 0} \frac{\Delta F_y}{\Delta A}.$$

Here, the subscript z specifies the orientation of a normal vector of the area ΔA.

9.3 Average Normal Stress in an Axially Loaded Bar

We consider the axially loaded bar shown on Figure 9.3.

Using the method sections one can quite easily show that the average stress σ is given by

$$\sigma = \frac{N}{A}.$$

where N is the internal resultant normal force (in this case $N = P$) and A is the cross sectional area.

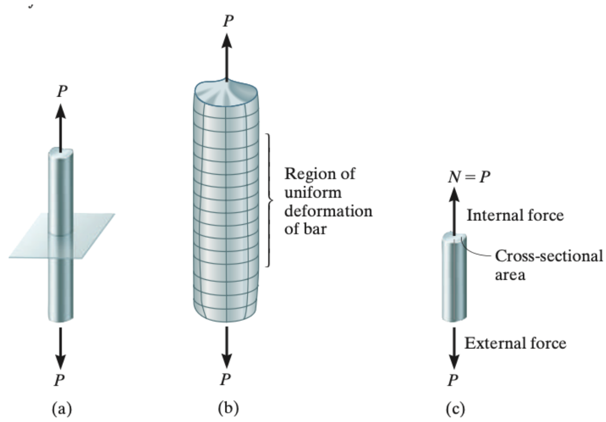

Figure 9.3: Axially loaded bar, deformed axially loaded bar and section of the axially loaded bar.

Lecture 11: Stress Design and Deformation 29. September 2025

9.4 Average Shear stress

Similarly it can be shown that the average shear stress τ_{avg} is given as

$$\tau_{\text{avg}} = \frac{V}{A}$$

where, V is the resultant internal shear force determined from the equations of equilibrium and A is the area of the section.

9.5 Allowable Stress Design

To ensure safety it is needed to restrict the allowed load on a structure to one that is less than what the structure can theoretically support. One method for doing this is to use the *factor of safety* (F.S.) which is a ratio between the failure load F_{fail} to the allowable load F_{allow},

$$\text{F.S.} = \frac{F_{\text{fail}}}{F_{\text{allow}}}.$$

Here F_{fail} is typically found through experimental testing of the material.

9.6 Strain

The normal strain has been defined in materials science as,

$$\epsilon_{\text{avg}} = \frac{\Delta L}{L}.$$

We recall that this is a dimensionless quantity.

Deformation, however, not only presents as normal strain. I.e. line segments of elements will not only expand and contract but also change direction. The *change in angle* between two segments, originally perpendicular, is referred to as the *shear strain*, denoted by γ and always measured in rad.

Lecture 12: Axially Loaded Members 2. Oktober 2025

10 Axial Load

10.1 Saint-Venant's Principle

In short, Saint-Venant's principle predicts that for an applied force P the distortion of a body will be large locally around the applied force but rather small everywhere else.

10.2 Elastic Dispalcement of an Axially Loaded Member

We wish to determine the relative displacement δ of one end of an axially loaded bar with respect to the other caused by axial loading. We neglect the localized concentrated deformations at the loadings with reference to Saint-Venant's principle.

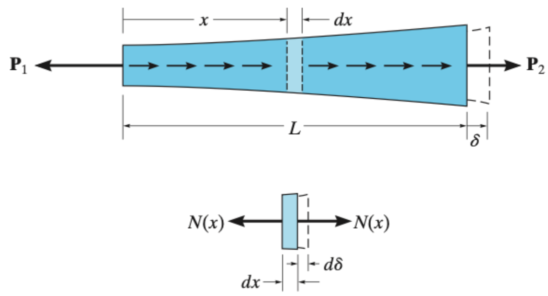

Figure 10.1: Axially loaded bar and differential element.

We consider the bar shown on Figure 10.1. Using the method of sections, we extract a differential element of length $\mathrm{d}x$ and cross sectional area $A(x)$ at position x where the modulus of elasticity is $E(x)$ and the internal axial force is $N(x)$. The stress and strain in the element will then be

$$\sigma = \frac{N(x)}{A(x)} \qquad \epsilon = \frac{\mathrm{d}\delta}{\mathrm{d}x}.$$

From Hooke's law the elongation can then be determined as

$$\delta = \int_0^L \frac{N(x)}{A(x)E(x)}\,\mathrm{d}x.$$

If the bar is assumed to have constant cross sectional area, and the material is assumed homogeneous and the external force is applied at each end the elongation is simply

$$\delta = \frac{NL}{AE}.$$

Lecture 14: Torsional Deformation 23. Oktober 2025

11 Torsion

11.1 Torsional Deformation of a Circular Shaft

Definition 8: Torque

Torque is a moment that tends to twist a member about its longtitudinal axis.

We consider a torque applied to a circular shaft made of a highly deformable material such as rubber. As torque is applied, the longtitudinal grid lines tend to distort into a helix as shown on Figure 11.1. Also, all the cross sections of the shaft will remain flat and all radial lines will remain straight and simply rotate during the deformation. For a small angle of twist the length of the shaft and its radius will remain unchanged.

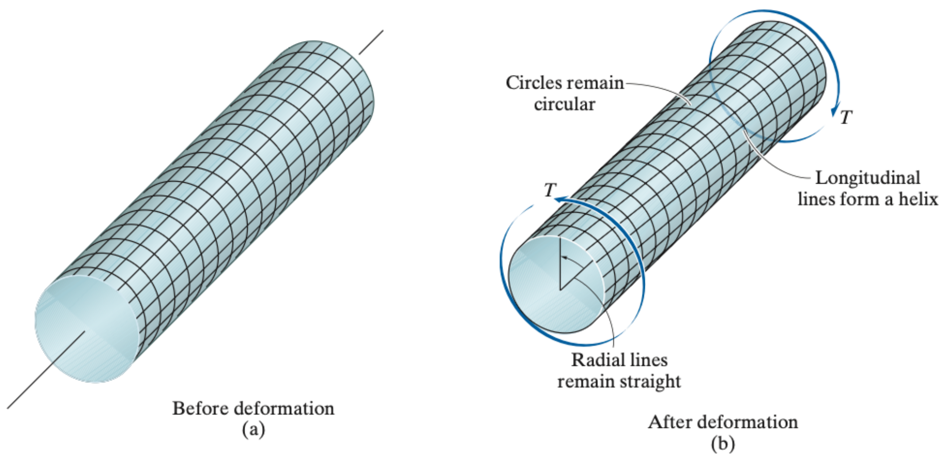

Figure 11.1: Distortion of grid lines in a highly deformable material due to an applied torque.

If we now fix one end of the shaft and apply a torque to the other end, then the undeformmed plane in Figure 11.2.a will distort into a skewed plane as shown.

A radial line, located on the cross section at distance x from the fixed end of the shaft, will rotate through an angle $\phi(x)$. This angle is called the *angle of twist*. As can be seen from the arguments to ϕ its magnitude depends on the position x along the shaft.

We will now consider a small disk-element, located at x as shown on Figure 11.2.b. Due to the deformation, the both the front and rear faces of the element will undergo rotation – the back face by $\phi(x)$ and the front face by an infinitesimally larger magnitude $\phi(x) + \mathrm{d}\phi$. The difference between these two rotations $\mathrm{d}\phi$ between the front and back face of the element causes a shear strain γ.

This angle (or shear strain) can be related to the angle $\mathrm{d}\phi$ by noting that for small angles $s = \theta r$, hence the

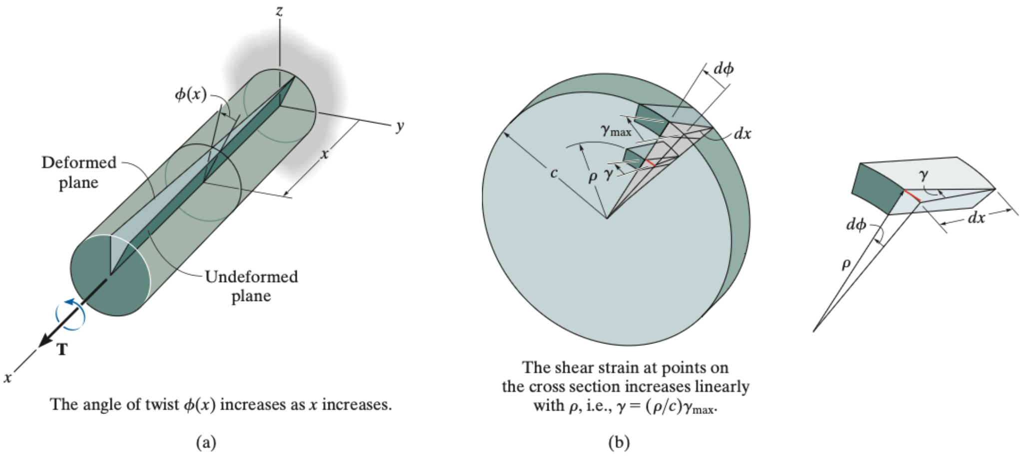

Figure 11.2: Torque applied to a shaft fixed at the other end.

length of the red arc in Figure 11.2.b is

$$\rho\,\mathrm{d}\phi = \mathrm{d}x\gamma \implies \gamma = \rho\frac{\mathrm{d}\phi}{\mathrm{d}x} \tag{1}$$

Since $\gamma_{\max}$ occurs for $\rho = c$, then $\mathrm{d}\phi/\mathrm{d}x = \gamma/\rho = \gamma_{\max}/c$ and therefore:

$$\gamma = \frac{\rho}{c}\gamma_{\max}.$$

I.e. the shear strain varies linearly along the radius of a cross section from zero at the center to a maximum $\gamma_{\max}$ at the outer edge.

11.2 The Torsion Formula

When an external torque is applied to a shaft it creates a corresponding internal torque T within the shaft. Here an equation that relates this internal torque to the shear stress distribution acting on the cross section of the shaft will be found.

If the material is linear elastic, then Hooke's law applies, i.e. $\tau = G\gamma$, or $\tau_{\max} = G\gamma_{\max}$. Consequently, a linear variation in shear strain leads to a linear variation in shear stress along the radius of a cross section. Hence, τ will vary from zero at the shafts longtitudinal axis to a maximum $\tau_{\max}$ at its outer edge.

$$\tau = \frac{\rho}{c}\tau_{\max} \tag{2}$$

On Figure 11.3, each element of the area, $\mathrm{d}A$, located at ρ is subjected to a force of

$$\mathrm{d}F = \tau\,\mathrm{d}A.$$

The torque produced by this force is

$$\mathrm{d}T = \rho\,(\tau\,\mathrm{d}A)\,.$$

For the entire cross section we require

$$T = \int_{\mathrm{A}} \rho\,(\tau\,\mathrm{d}A) = \int_{\mathrm{A}} \rho\frac{\rho}{c}\tau_{\max}\,\mathrm{d}A.$$

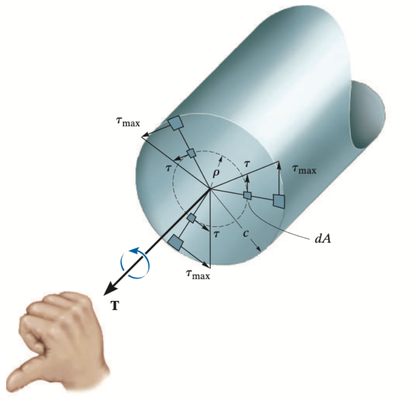

Figure 11.3: Linear variation of shear stress along a radial line.

Since $\tau_{\max}/c$ is constant we can rewrite this as:

$$T = \frac{\tau_{\max}}{c} \int_{\mathrm{A}} \rho^2 \,\mathrm{d}A.$$

This integral represents the polar moment of inertia of the circular cross section about the shaft's longtitudinal axis. This is symbolized as J. As a result the above equation can be rewritten as:

$$\tau_{\max} = \frac{Tc}{J} \tag{3}$$

Where:

$\tau_{\max}$	The maximum shear stress within the shaft, which occurs at its outer surface
T	The resultant *internal torque* acting at the cross section.
J	The polar moment of inertia of the cross-sectional area
c	The outer radius of the shaft .

We can insert Equation (3) into Equation (2) to get the shear stress at the intermediate distance ρ on the cross section as

$$\tau = \frac{T\rho}{J}.$$

Either of the above two equation are referred to as the torsion formula. This is only applicable if the shaft has a circular cross section, the material is homogeneous, and behaves in a linear elastic manner.

11.3 Power Transmission

Shafts and tubes with circular cross sections are often used to transmit power from a motor. The work transmitted by a rotating shaft is equal to the torque applied times the angle of rotation. Therefore, if during an instant of time $\mathrm{d}t$ an applied torque $\vec{T}c$ causes the shaft to rotate $\mathrm{d}\theta$, then the work done is $T\,\mathrm{d}\theta$ and the instantaneous power is

$$P = T\frac{\mathrm{d}\theta}{\mathrm{d}t}.$$

The shaft's angular velocity is $\omega = \mathrm{d}\theta/\mathrm{d}t$, and hence the power becomes:

$$P = T\omega.$$

For motors the frequency of a shaft's rotation f is often reported. Using this the power is

$$P = 2\pi f T.$$

Lecture 15: Statically Indeterminate Torque-loaded Members 27. Oktober 2025

11.4 Angle of Twist

We consider a shaft with a circular cross section that can vary gradually along its length as shown on Figure 11.4.a. Furthermore, the material is assumed to be homogeneous and linear elastic. We neglect the localized deformation that occur at points of applications of the torque and there the cross section changes abruptly.

Using the method of sections, a differential disk of thickness $\mathrm{d}x$, located at position x is isolated from the shaft as shown on Figure 11.4.b. At this point the internal torque is $T(x)$. Due to $T(x)$, the disk will twist, such that the relative rotation of one of its faces with respect to the other $\mathrm{d}\phi$. As a result an element of the material located at an arbitrary radius ρ within the disk will undergo a shear strain γ. The values of γ and $\mathrm{d}\phi$ are related by Equation (1) as

$$\mathrm{d}\phi = \gamma \frac{\mathrm{d}x}{\rho}.$$

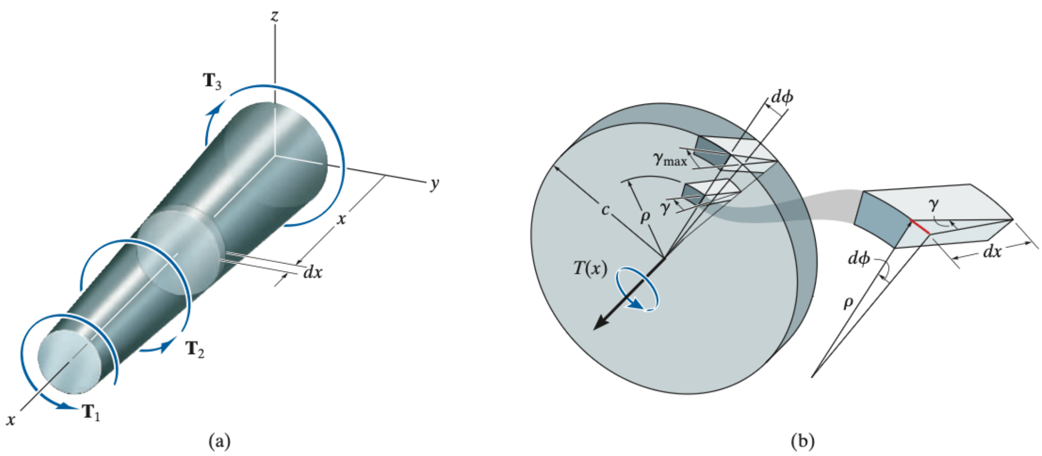

Figure 11.4: Shaft subjected to a torque.

Since Hooke's law applies and the shear stress can be expressed in terms of the applied torque using the torsion formula, we can find the angle of twist as

$$\mathrm{d}\phi = \frac{T(x)}{J(x)G(x)}\,\mathrm{d}x.$$

Integrating over the entire length L of the shaft we get the angle of twist of the entire shaft as

$$\phi = \int_0^L \frac{T(x)}{J(x)G(x)}\,\mathrm{d}x.$$

11.4.1 Constant Torque and Cross-sectional Area

For a homogeneous material G is constant. Furthermore, if the cross-sectional area and external torque along the length of the shaft is also constant, the integral above reduces to

$$\phi = \frac{TL}{JG}.$$

11.4.2 Sign Convention

By convention, the right-hand rule is applied with the thumb outwards from the shaft with the fingers curling in the direction of positive torque T.

11.5 Statically Indeterminate Torque-loaded Members

A torsionally loaded shaft is statically indeterminate, if the moment equation of equilibrium, about the axis of the shaft, is not adequate to determine the unknown torques acting on the shaft. An example of this is shown on Figure 11.5.a. As shown on the free-body diagram of Figure 11.5.b the reactive torques at the fixed supports A and B are unknown. Along the axis of the shaft, we require

$$\sum M = 0 \implies \qquad 500\,\mathrm{N\,m} - T_A - T_B = 0.$$

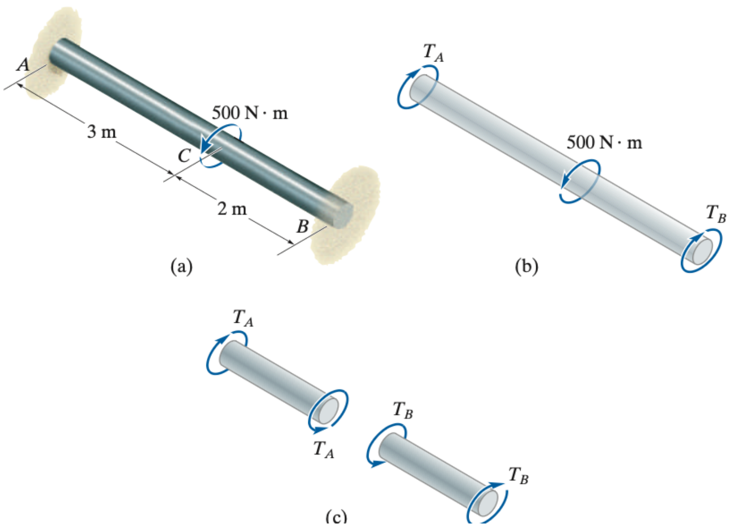

Figure 11.5: Statically indeterminate torque-loaded member.

In order to obtain a solution we will use the fact that the angle of twist of one end of the shaft with respect to the other end is equal to zero singe the end support are fixed. Therefore,

$$\phi_{A/B} = 0.$$

Provided the material is homogeneous and linear elastic, we can then apply the load displacement relation $\phi = TL/JG$ to express the equation in terms of the unknown torques. Realizing the internal torque in segment AC is T_A and in segment CB it is $-T_B$ as shown on Figure 11.5.c. We get:

$$\frac{T_A\,(3\,\mathrm{m})}{JG} - \frac{T_B\,(2\,\mathrm{m})}{JG} = 0$$
$$\implies \qquad T_A = 200\,\mathrm{N\,m} \qquad \text{and} \qquad T_B = 300\,\mathrm{N\,m}.$$

Lecture 16: Bending – Shear and Moment Diagram 30. Oktober 2025

12 Bending

12.1 Internal Loading as a Function of Position

Definition 9: Beams

A beam is a slender member that supports a load perpendicular to its longitudinal axis.

In general, beams are long, straight members having a cross-sectional area. Oftentimes these are classified based on how they are supported. A few different support types are shown on Figure 12.1.

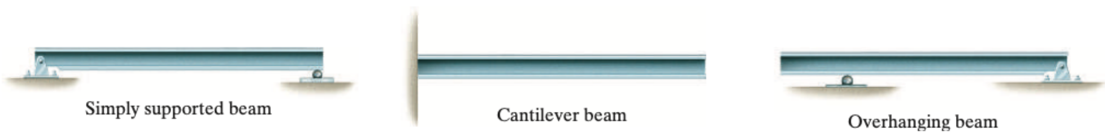

Figure 12.1: Basic support types for beams.

Due to the applied loadings, beams develop an internal shear force and bending moment that, in general, varies from point to point along the axis of the beam and therefore it is important to determine the *maximum* shear and moment in the beam. One way to do this is to express V and M as functions of their arbitrary position x along the beam's axis and then plot these functions. These plots are called the shear and moment diagrams, respectively. The maximum values of V and M can be directly obtained from these diagrams. Also since these provide an easily identifiable visual cue for the variation in V and M they are often used to determine where supports or reinforcements should be added.

In cases there the beam supports several distributed loads, concentrated forces, and/or couple moments, the internal shear and moment functions of x will be discontinuous at the points where the loads are applied. Because of this, these functions must be determined for each region of the beam *between* any two discontinuities of loading.

12.1.1 Beam Sign Convention

The "normal" sign convention of beam loading is shown on Figure 12.2. The positive *distributed load* acts *upward*, the positive *internal shear force* causes a *clockwise* rotation of the beam segment on which it acts, and the positive *moment* causes *compression* in the *top* of the segment such that it bends the segment in a way that "holds water". Opposite loadings are considered negative.

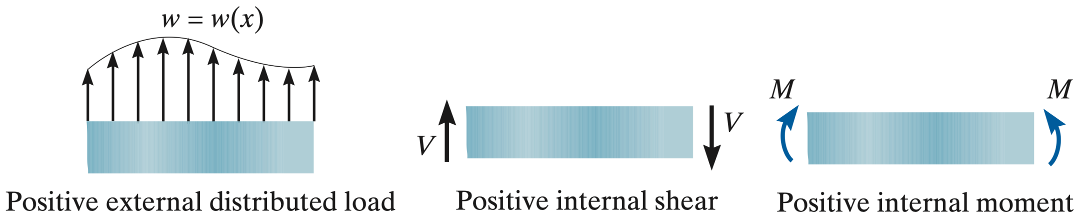

Figure 12.2: Beam Sign convention.

12.2 Graphical Method for Constructing Shear and Moment Diagrams

We consider the beam shown on Figure 12.3.a which is subject to an arbitrary distributed loading. A free-body diagram for a small segment Δx of the beam is depicted on Figure 12.3.b. This segment is chosen at position x where there is no concentrated force or couple moment.

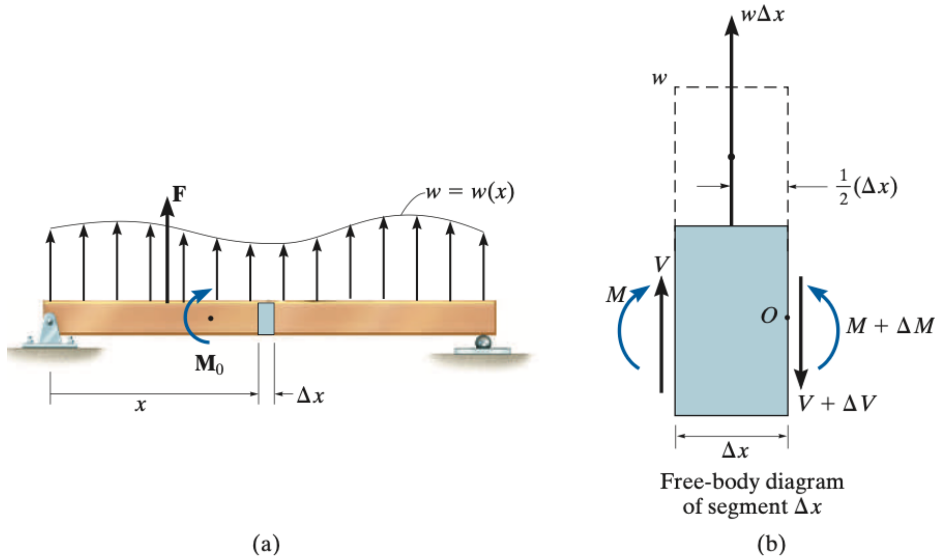

Figure 12.3: Distributed load on a beam.

Notice that all the loadings shown on the segment act in their positive directions according to the sign convention established in Section 12.1.1. Both the internal resultant shear and moment acting on the right face must be increased by a small amount to maintain equilibrium. The distributed load (which is approximately constant over Δx) has been replaced by a resultant $w\Delta x$ that acts at the middle of the segment ($1/2\Delta x$ from either side). The equations of equilibrium gives us

$$\begin{aligned}
\sum F_y &= 0 \\
V0w\Delta x - (V + \Delta V) &= 0 \\
\Delta V &= w\Delta x \\
\sum M_O &= 0 \\
-V\Delta x - M - w\Delta x\left(\frac{1}{2}\Delta x\right) + (M + \Delta M) &= 0 \\
\Delta M &= V\Delta x + w\frac{1}{2}\left(\Delta x\right)^2 .
\end{aligned}$$

Dividing by Δx and taking the limit as $\Delta x \to 0$ the above two equations become:

$$\begin{aligned}
\overbrace{\frac{\mathrm{d}V}{\mathrm{d}x}}^{\text{Slope of shear diagram}} &= \overbrace{w}^{\text{Distributed loading}} \\
\underbrace{\frac{\mathrm{d}M}{\mathrm{d}x}}_{\text{Slope of moment diagram}} &= \underbrace{V}_{\text{Shear}} .
\end{aligned}$$

The first of these two equations states that at any point the *slope* of the shear diagram equals the intensity of the distributed loading. In a similar manner the second states that the slope of the moment diagram is equal to the shear. These may also be rewritten as

$$\begin{aligned}
\Delta V &= \int w\,\mathrm{d}x \\
\Delta M &= \int V\,\mathrm{d}x.
\end{aligned}$$

Lecture 17: Bending – Deformation of Beams

3. November 2025

12.3 Bending Deformation of a Straight Member

On Figure 12.4.a square cross section of an undeformed member is shown. When a bending moment is applied it tends to distort these lines as shown on Figure 12.4.b.

The horizontal lines become curved whilst the vertical lines remain straight but rotate. A bending moment causes the material within the bottom of the member to stretch, whilst the top compresses. Consequently, between these two regions there must be a surface called the *neutral surface*, in which horizontal fibers of the material will not undergo a change in length. As noted on Figure 12.5 the axis that lies along the neutral surface is called the neutral axis

We now focus on the small member depicted on Figure 12.6.a located at a distance x along the beam's length. This element is shown in its deformed state on Figure 12.6.b. Here, the line segment Δx, located on the neutral surface, does not change length, whereas any line segment Δs, located at an arbitrary distance

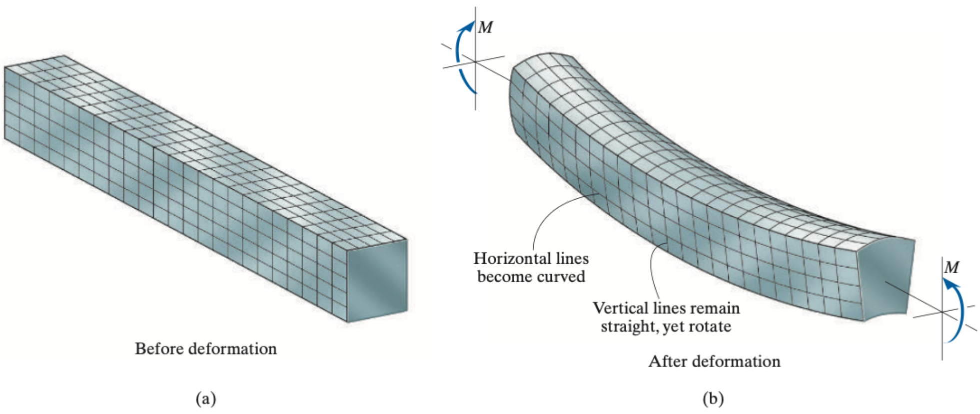

Figure 12.4: Bending deformation of a straight member.

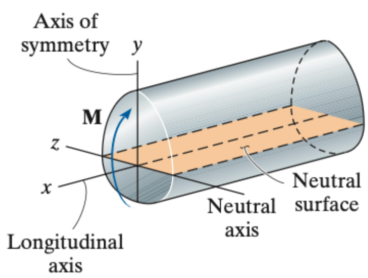

Figure 12.5: Neutral surface and neutral axis of a member.

above the neutral surface contracts and becomes $\Delta s'$ after the deformation. The normal strain along Δs is determined as

$$\epsilon = \lim_{\Delta s \to 0} \frac{\Delta s' - \Delta s}{\Delta s}.$$

We can also represent this strain in terms of the location y of $\Delta s'$ and the radius of curvature ρ of the longitudinal axis of the element. We get:

$$\epsilon = \lim_{\Delta \theta \to 0} \frac{(\rho - y)\,\Delta\theta - \rho\Delta\theta}{\rho\Delta\theta} = -\frac{y}{\rho}.$$

Since $1/\rho$ is constant at x the above equation indicates that the longitudinal normal strain will vary linearly with y measured from the neutral axis. A contraction $(-\epsilon)$ will occur in fibers located above the neutral axis $(+y)$ and vice versa. The maximum strain will occur at the outermost fiber, located at a distance of $y = c$ from the neutral axis. Since $\epsilon_{\max} = c/\rho$ then

$$\frac{\epsilon}{\epsilon_{\max}} = -\left(\frac{y/\rho}{c/\rho}\right)$$

so that

$$\epsilon = -\frac{y}{c}\epsilon_{\max}.$$

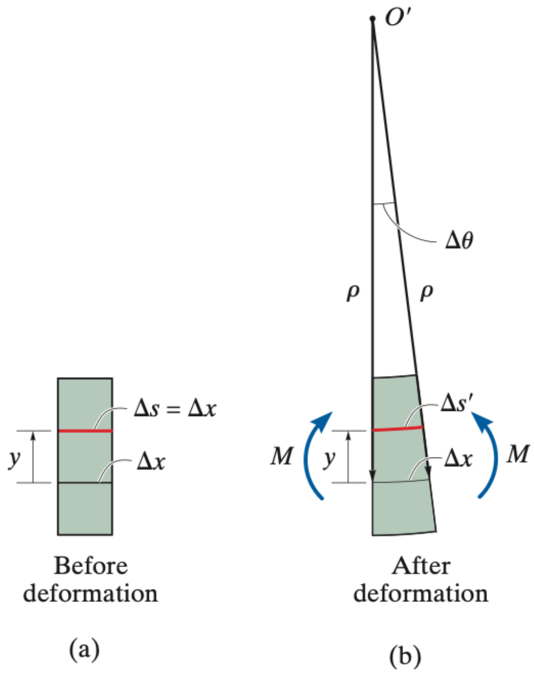

Figure 12.6: Bending moment on a small element

12.4 The Flexure Formula

We consider a linear elastic material, meaning Hooke's law is applicable and results in a linear variation of normal strain. Hence, the normal strain, σ will vary from zero at the neutral axis to a maximum value $\sigma_{\max}$, a distance c farthest from the neutral axis. We get

$$\sigma = -\frac{y}{c}\sigma_{\max}.$$

12.4.1 Location of the Neutral Axis

To locate the position of the neutral axis, we require the resultant force produced by the stress distribution acting over the cross sectional area to be zero. Based on Figure 12.7, we get

$$\begin{aligned} F_R &= \sum F_x \\ 0 &= \int_{\text{A}} \mathrm{d}F \\ &= \int_{\text{A}} \sigma \,\mathrm{d}A \\ &= \int_{\text{A}} -\frac{y}{c}\sigma_{\max}\,\mathrm{d}A \\ &= -\frac{\sigma_{\max}}{c}\int_{\text{A}} y\,\mathrm{d}A \\ \implies \quad 0 &= \int_A y\,\mathrm{d}A. \end{aligned}$$

I.e. the "first moment" of the cross-sectional area about the neutral axis must be zero. Therefore the neutral axis must also be the horizontal centroidal axis for the cross section.

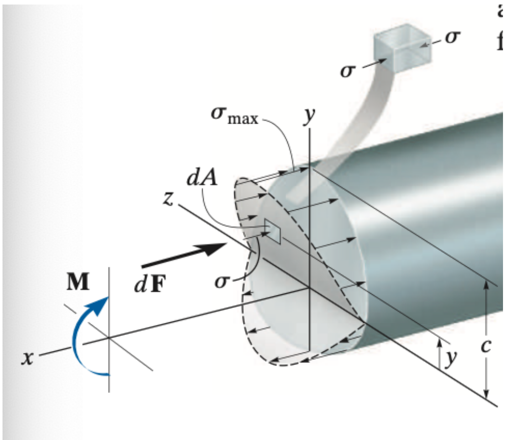

Figure 12.7: Bending stress variation

12.4.2 Bending Moment

If we require the moment M to be equal to the moment produced by the stress distribution over the neutral axis, the stress in the beam can be found. I.e.:

$$\begin{aligned}(M_R)_z &= \sum M_z \\ M &= \int_{\mathrm{A}} y \,\mathrm{d}F \\ &= \int_{\mathrm{A}} y \,(\sigma \,\mathrm{d}A) \\ &= \int_{\mathrm{A}} y \frac{y}{c} \sigma_{\max} \,\mathrm{d}A \\ \implies \quad M &= \frac{\sigma_{\max}}{c} \int_{\mathrm{A}} y^2 \,\mathrm{d}A.\end{aligned}$$

This integral represents the *moment of inertia* of the cross sectional area about the neutral axis. Hence, symbolizing this as I it can be rewritten as

$$\sigma_{\max} = \frac{Mc}{I} \implies \sigma = -\frac{My}{I}.$$

Lecture 18: Deflection of Beams

6. November 2025

12.5 Unsymmetric Bending

When developing the Flexure formula above it was required that the cross-sectional area was symmetric about an axis perpendicular to the neutral axis and the resultant moment $\vec{M}$ to act around the neutral axis. Now a general formula will be derived

12.5.1 Moment applied about Principal Axis

We consider the beam's cross section to have the unsymmetrical shape shown on Figure 12.8.a. The coordinate system is placed such that the origin is located at the centroid C of the cross section and the resultant moment $\vec{M}$ acts around the $+z$ axis. The shear stress distribution over the entire cross sectional area must have a zero force resultant:

$$0 = -\int_{\mathrm{A}} \sigma \, \mathrm{d}A,$$

the moment of the stress distribution about y must be 0:

$$0 = -\int_{\mathrm{A}} z\sigma \, \mathrm{d}A,$$

and the moment about the z-axis must equal $\vec{M}$:

$$M = \int_{\mathrm{A}} y\sigma \, \mathrm{d}A.$$

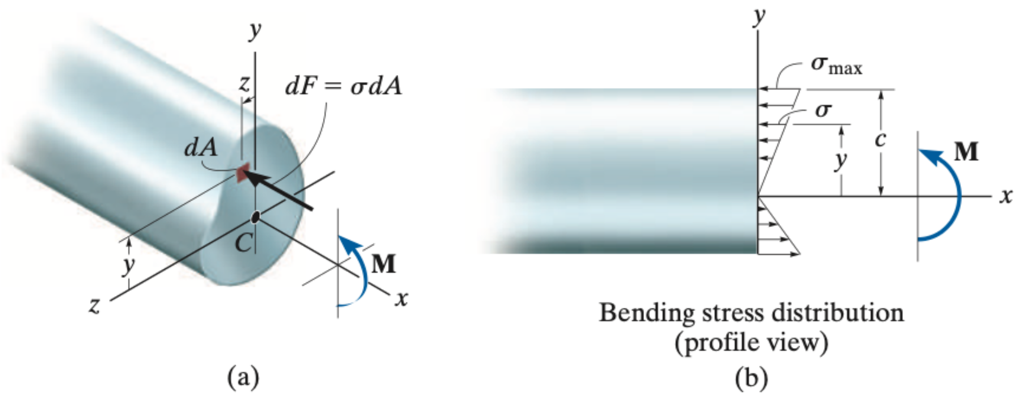

Figure 12.8: Moment applied about the principal axis of a beam

The first of these equations is satisfied since the z-axis passes through the centroid of the area. Also, as the z-axis represents the neutral axis for the cross section. Therefore:

$$0 = \frac{-\sigma_{\max}}{c} \int_{\mathrm{A}} yz \, \mathrm{d}A$$

which requires

$$\int_{\mathrm{A}} yz \, \mathrm{d}A = 0.$$

The above integral is called the *product of inertia* for the area. This will always be zero so long as the y and z axes are chosen as the principal axes of inertia for the area.

12.5.2 Moment Arbitrarily Applied

Sometimes a member will be loaded such that $\vec{M}$ does not act about one of the principal axes of the cross section. When this occurs, the moment should first be resolved into components along the principal axes, then the flexure formula can be used to determine the normal stress caused by each component. Finally, the principle of superposition can be applied to find the resultant normal stress at the point.

That is:

$$\sigma = -\frac{M_z y}{I_z} + \frac{M_y z}{I_y} \tag{4}$$

12.5.3 Orientation of the Neutral Axis

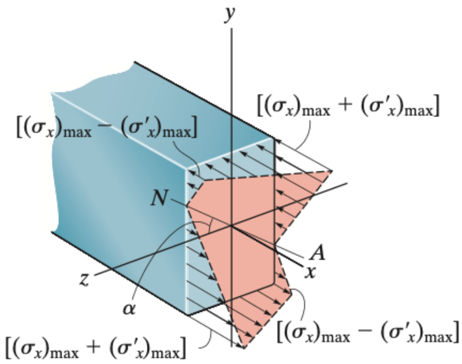

Figure 12.9: The orientation of the neutral axis.

The equation defining the neutral axis and its inclination α, from Figure 12.9, can be determined by applying Equation (4) to a point $0, y, z$ where $\sigma = 0$, which gives:

$$y = \frac{M_y I_z}{M_z I_y} z.$$

Since $M_z = M\cos\theta$ and $M_y = M\sin\theta$ then

$$y = \left(\frac{I_z}{I_y}\tan\theta\right) z.$$

The slope of the neutral axis is $\tan\alpha = y/z$, hence:

$$\tan\alpha = \frac{I_z}{I_y}\tan\theta.$$

13 Deflection of Beams and Shafts

13.1 The Elastic Curve

We consider the beam shown on Figure 13.1.a and remove the small element located at distance x from the left end and having an undeformed length of $\mathrm{d}x$. On Figure 13.1.b the localized y-coordinate is measured from the elastic curve (neutral axis) to the fiber in the beam that has an original length of $\mathrm{d}s = \mathrm{d}x$ and a deformed length $\mathrm{d}s'$.

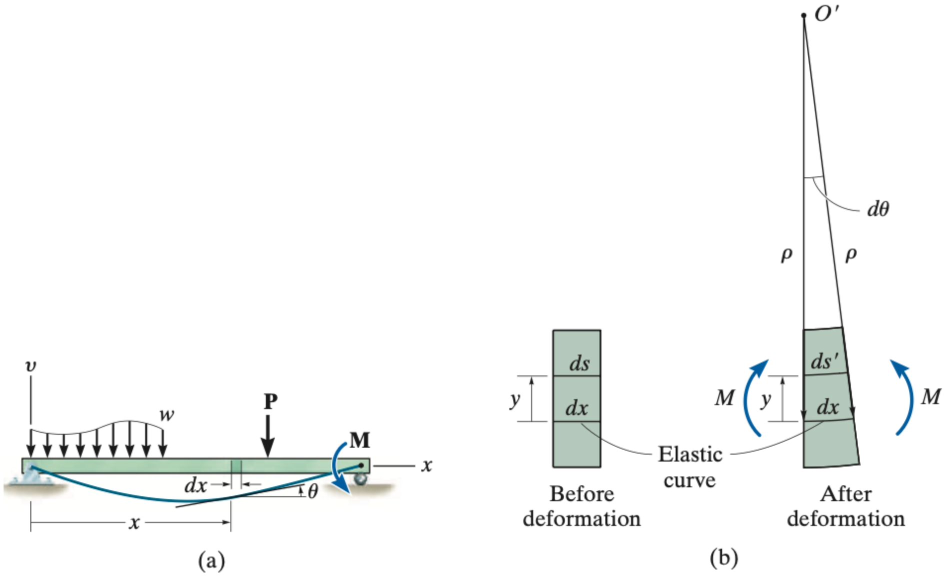

Figure 13.1: Moment-curvature relationsip in a beam.

From Hooke's law we can obtain

$$\frac{1}{\rho} = \frac{M}{EI}.$$

Where:

ρ	The radius of curvature at the point
M	The internal moment in the beam at the point
E	The modulus of elasticity of the material
I	The moment of inertia about the neutral axis of the beam

13.2 Slope and Displacement by Integration

For most problems the *flexural rigidity* (EI) is constant along the length of the beam. Assuming this is the case we obtain:

$$EI\frac{\mathrm{d}^4 v}{\mathrm{d}x^4} = w(x)$$
$$EI\frac{\mathrm{d}^3 v}{\mathrm{d}x^3} = V(x)$$
$$EI\frac{\mathrm{d}^2 v}{\mathrm{d}x^2} = M(x).$$

13.2.1 Boundary Conditions

Solving any of these equations requires successive integrations to obtain v. For each integration, a constant of integration is introduced and one must therefore solve for all these constants to obtain a unique solution for a particular problem.

Lecture 19: Superposition

10. November 2025

13.3 Method of Superposition

The differential equation $EI\,\mathrm{d}^4v/\mathrm{d}x^4 = w(x)$ satisfies the two necessary requirements for applying the principle of superposition. These are that the load is assumed linearly related to the displacement, and the load is assumed not to significantly change the original geometry of the beam or shaft. As a result, one may superimpose a series of separate loadings acting on a beam. E.g. if v_1 is the total displacement for one load and v_2 is the displacement for another load, the total displacement for both loads acting together is simply the algebraic sum $v_1 + v_2$.

Lecture 20: Statically Indeterminate Beams

13. November 2025

13.4 Statically Indeterminate Beams and Shafts – Method of Superposition

In order to use the method of superposition to solve for the reactions on a statically indeterminate beam, we must first identify the redundants and remove them from the beam. The will produce the *primary beam*, which will be statically determinate and stable. Using superposition, we add a succession of similarly supported beams, each loaded with a separate redundant to the primary beam. The redundants may be determined from the *conditions of compatibility* that exist at each support where a redundant acts.

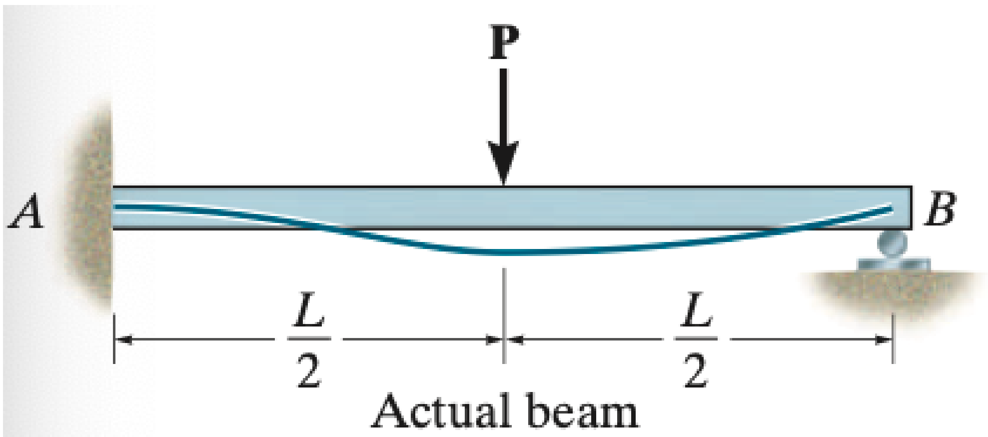

Figure 13.2:

To clarify, consider the beam depicted on Figure 13.2. If we choose the reaction $\vec{B}_y$ at the roller as our redundant, then the primary beam becomes as shown on Figure 13.3 and the beam with only the redundant $\vec{B}_y$ acting on it is shown on Figure 13.4. The displacement at the roller is set to be zero, and since the displacement of B on the primary beam is v_B and $\vec{B}_y$ causes B to be displaced upward v'_B, we can write the compatibility equation as:

$$0 = -v_B + v'_B.$$

These displacements can be expressed in terms of the loads using the table in Appendix D of the book. These load-displacement relations are

$$v_B = \frac{5PL^3}{48EI} \qquad v'_B = \frac{B_y L^3}{3EI}.$$

Substituting these into the compatibility equations we get

$$0 = -\frac{5PL^3}{48EI} + \frac{B_y L^3}{3EI}$$
$$B_y = \frac{5}{16}P.$$

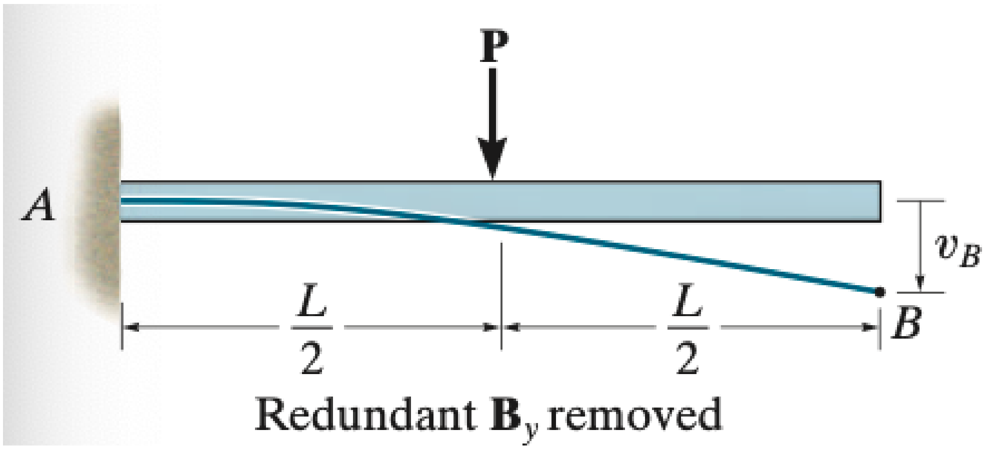

Figure 13.3:

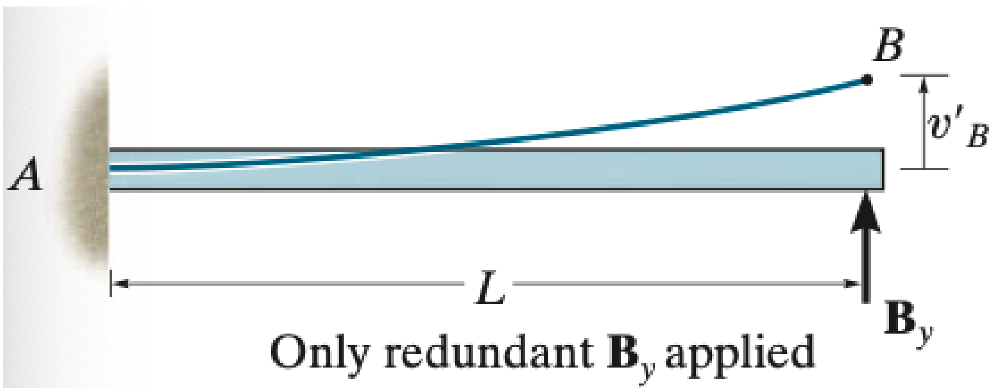

Figure 13.4:

Now that $\vec{B}_y$ is known, the reactions at the wall can be determined from the equilibrium equations. The choice of the redundant is arbitrary, provided the beam remains stable – e.g. the moment at A on Figure 13.2 could also have been chosen as a redundant.

Lecture 21: Transverse Shear 17. November 2025

14 Transverse Shear

14.1 Shear in Straight Members

In general, a beam will support both an internal shear and a moment. The shear $\vec{V}$ is the resultant of transverse shear-stress distribution acting over the cross section of the beam as shown on Figure 14.1. Due to the complementary property of shear, this stress will also create a corresponding longitudinal shear stress that will act over the length of the beam.

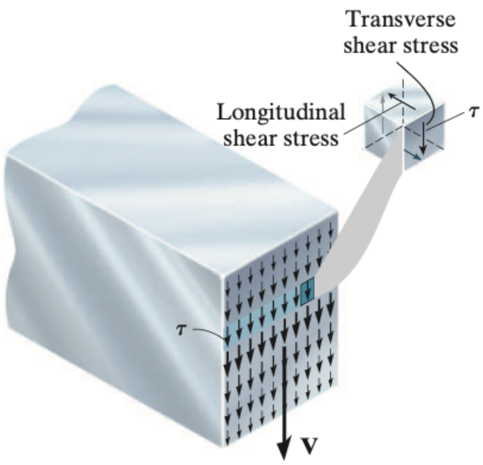

Figure 14.1: Transverse and longitudinal shear stress.

To illustrate this, we consider the beam made of the three boards shown on Figure 14.2a. If the top and bottom surfaces of each board are smooth and the boards are not bonded together, then the application of the load $\vec{P}$ will cause the boards to slide relative to each other. If the boards instead are bonded together, then the longitudinal shear stress acting in the bonding between the boards will prevent their relative sliding as shown on Figure 14.2b.

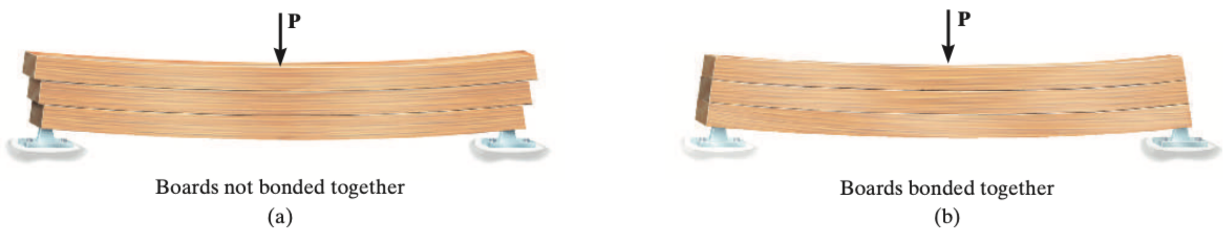

Figure 14.2: Effect of longitudinal shear stress.

As a result of the shear stress, shear strains will be induced that will tend to distort the cross section in a rather complex manner.

14.2 The Shear Formula

As the strain distribution for shear is not easily defined, as in the case of axial load, torsion, or bending, we will have to obtain the shear-stress distribution in an indirect manner. We will first consider the horisontal force equilibrium of a portion of an element with length dx taken from the beam as shown on Figure 14.3a.

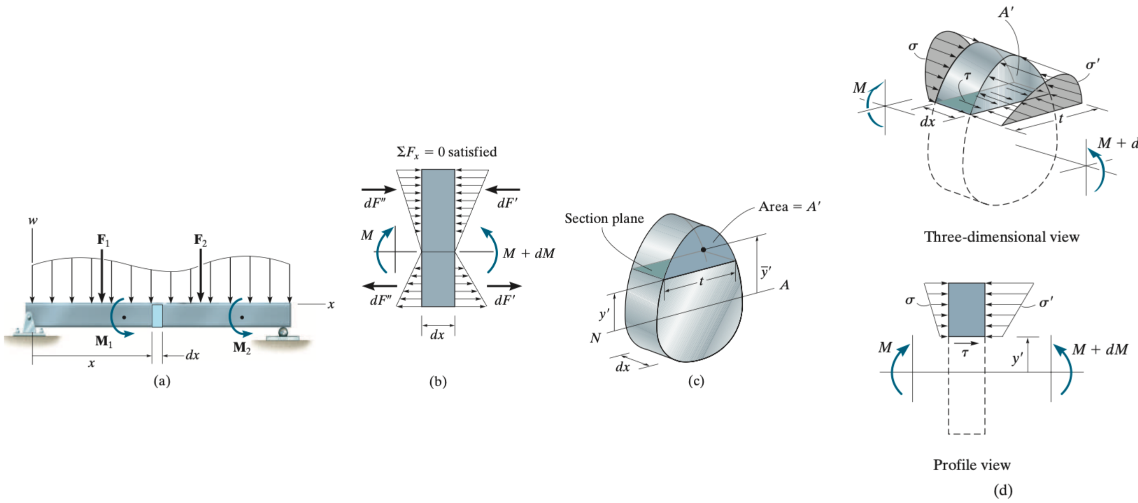

Figure 14.3: Infinitesimal element of the beam.

A free body diagram of this element is depicted on Figure 14.3b. The normal shear stress distribution acting on it is caused by the bending moments M and $M + \mathrm{d}M$. We have excluded the effects of V, $V + \mathrm{d}V$ and $w(x)$ since these loadings are all vertical and will therefore not have a horizontal force component. Here $\sum F_x = 0$ is automatically satisfied since the stress distribution on each side of the element forms only a couple moment and therefore no force resultant is induced.

We now consider the top portion of the element sectioned at y' from the neutral axis, depicted on Figure 14.3c. It is on this bottom sectioned plane that we want to find the horizontal shear stress. The top segment has width t at the section, and the two cross-sectional sides each have an area A'. The segment's free body diagram is shown on Figure 14.3d. As the resultant moment on each side of the element differ by dM, the condition $\sum F_x = 0$ is not satisfied unless a longitudinal shear stress τ acts over the bottom sectioned plane. To simplify the analysis, we assume that the shear stress is constant across the width t of the bottom plane. To find the horizontal force created by the bonding moments, we will assume that the effect of warping due to shear is small, so that it can be neglected. This assumption is particularly true for the common case os a slender beam.

Using the flexure formula we have:

$$-\int_{A'} \sigma' \,\mathrm{d}A' + \int_{A'} \sigma \,\mathrm{d}A' + \tau(t\,\mathrm{d}x) = 0$$

$$-\int_{A'} \left(\frac{M + \mathrm{d}M}{I}\right) y \,\mathrm{d}A' + \int_{A'} \left(\frac{M}{I}\right) y \,\mathrm{d}A' + \tau(t\,\mathrm{d}x) = 0$$

$$\left(\frac{\mathrm{d}M}{I}\right) \int_{A'} y \,\mathrm{d}A' = \tau(t\,\mathrm{d}x). \tag{5}$$

Solving for τ, we obtain

$$\tau = \frac{1}{It}\frac{\mathrm{d}M}{\mathrm{d}x}\int_{A'} y\,\mathrm{d}A'.$$

Here $V = \mathrm{d}M/\mathrm{d}x$. Also, the integral represents the moment of the area A' about the neutral axis, which we will denote by the symbol Q. Since the location of the centroid area of A' is determined from $\overline{y}' = \int_{A'} y\,\mathrm{d}A'/A'$, we can write the above as:

$$Q = \int_{A'} y\,\mathrm{d}A' = \overline{y}'A'. \tag{6}$$

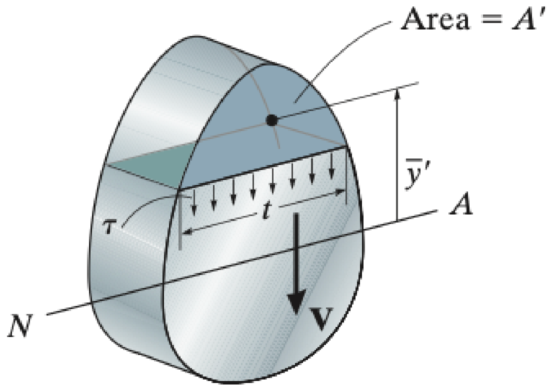

Figure 14.4:

Substituting, we get the so-called shear formula:

$$\tau = \frac{VQ}{It}. \tag{7}$$

With reference to Figure 14.4, τ is the shear stress in the member located at y' from the neutral axis, V is the shear force, I is the moment of inertia of the entire cross-sectional area calculated about the neutral axis, t is the width of the member's cross section, meassured at y', and $Q = \overline{y}'A'$, where A' is the area of the top portion of the members cross section at plane y' where t is measured, and $\overline{y}'$ is the distance from the neutral axis to the centroid of A'.

Calculating Q Of all the variables in the shear formula, Equation (7), Q is usually the most difficult to define. This represents the *moment about the neutral axis of the cross-sectional area that is above or below the level where the shear stress is to be determined*. Is is this area A' that is "held onto" the rest of the beam by the longitudinal shear stress as the beam undergoes bending, Figure 14.3d.

14.3 Shear Flow in Built-up Members

Occasionally in engineering practice, beams are constructed from several composite parts in order to achieve a greater resistance to loading. When the loads cause the members to bend, a fastener is needed to keep the members from sliding against each other as shown on Figure 14.2. In order to design these, it is necessary to know the shear force they must resist. This loading, when measured as a force per unit length of the beam is referred to as a shear flow q.

The magnitude of the shear flow is obtained using a procedure similar to the one used for finding the shear stress in the beam. Doing this one obtains the shear flow formula:

$$q = \frac{VQ}{I}$$

where, q is the shear flow, V is the shear force, I is the moment of inertia of the entire cross-sectional area about the neutral axis and $Q = \overline{y}'A'$ where A' is the cross sectional area that is connected to the beam at the juncture where the shear flow is calculated and $\overline{y}'$ is the distance from the neutral axis to the centroid A'.

Fastener Spacing When segments of a beam are connected by fasteners, such as nails of bolts, their spacing s along the beam can easily be determined, For example, consider a fastener that can support a maximum shear force of F before it fails. If these nails are used to construct a simple beam by combining two boards, then the nails must resist the shear flow q, between the boards. The nail spacing is therefore:

$$F = q \cdot s.$$

Lecture 22: Combined Loading 20. November 2025

15 Combined Loadings

15.1 Thin-Walled Pressure Vessels

The stresses acting in a cylindrical or spherical pressure vessel can easily be analyzed provided it has a *thin wall* – i.e. an inner-radius-to-wall–thickness ratio of 10 or more. Specifically, when $r/t = 10$ the results of a thin-wall analysis will predict a stress that is approximately 4% less than the actual maximum stress in the vessel. For larger r/t ratios this error will be even smaller.

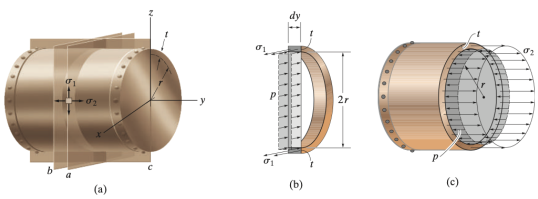

Figure 15.1: Thin-walled cylindrical vessel.

15.1.1 Cylindrical vessels

The cylindrical vessel on Figure 15.1a has a wall thickness t, inner radius r and is subjected to internal gas pressure p. To find the *circumferential* or *hoop stress* we will section the vessel by planes a, b, and c. A free-body diagram of the back segment along with its contained gas is shown on Figure 15.1b. For simplicity, only the loadings in the x direction are shown. These are caused bu the uniform hoop stress σ_1 acting on the vessel's wall, and the pressure acting on the vertical face of the gas. For equilibrium in the x-direction we require

$$2\left(\sigma_1(t\,\mathrm{d}y)\right) - p(2r\,\mathrm{d}y) = 0 \implies \sigma_1 = \frac{pr}{t}.$$

The longitudinal stress can be determined by considering the left portion of section b on Figure 15.1a. As shown on the free-body diagram on Figure 15.1c, σ_2 acts uniformly throughout the wall, and p acts on the section of the contained gas. Since the mean radius is approximately equal to the vessels inner radius, equilibrium in the y direction requires:

$$\sigma_2(2\pi rt) - p\left(\pi r^2\right) = 0 \implies \sigma_2 = \frac{pr}{2t}.$$

Where σ_1 and σ_2 are the normal stress in the hoop and longitudinal directions, respectively, p is the internal gauge pressure, r is the inner radius and t the wall thickness.

By comparison, we see that the hoop stress is twice as large as the longitudinal stress.

15.1.2 Spherical Vessels

For a spherical vessel, the analysis is rather similar. We section the vessel on Figure 15.2a in half as shown, which leaves us with the free-body diagram of Figure 15.2b. Equilibrium in the y direction requires

$$\omega_2\left(2\pi rt\right) - p\left(\pi r^2\right) = 0 \implies \sigma_2 = \frac{pr}{2t}.$$

This is the same as the longitudinal stress for a cylindrical vessel.

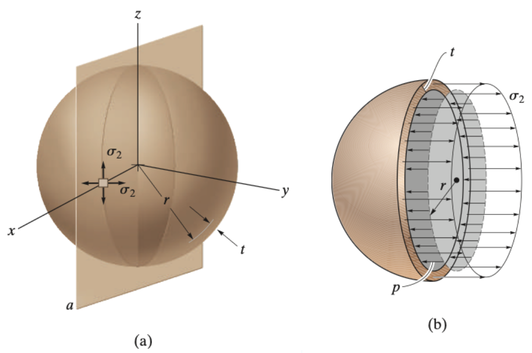

Figure 15.2: Thin-walled spherical vessel

Lecture 24: Plane stress transformation 27. November 2025

16 Stress and Strain Transformation

16.1 Plane-Stress Transformation

The general stress state at a point is characterized by six normal and shear stress components as depicted on Figure 16.1a. This state of stress is, however, not often encountered in actuality. Instead, most loadings are coplanar and can be analyzed in a *single plane*. When this is the case, the material is said to be subject to *plane stress*.

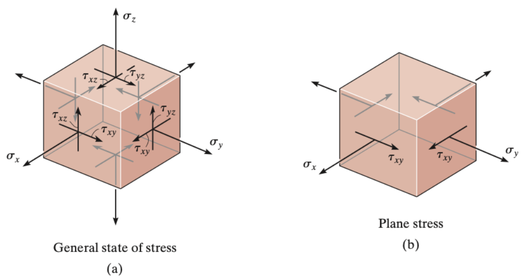

Figure 16.1: State of stress at a general point.

The general state of plane stress at a point is shown on Figure 16.1b. This is represented by a combination of two normal stress components, σ_x, σ_y, and one shear stress component τ_{xy}, which act on only four faces of the element as shown on Figure 16.2a. It is worth realizing, that this state of stress is produced on an element having a different orientation θ as in Figure 16.2b, then it will be subjected to three different stress components, $\sigma_{x'}$, $\sigma_{y'}$, and $\tau_{x'y'}$, measured relative to the x', y' axes.

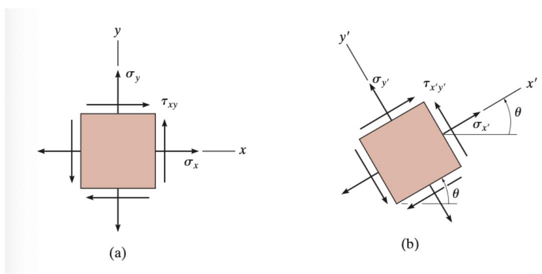

Figure 16.2: General state of plane stress.

16.2 General Equations of Plane Stress Transformation

The method of transforming the normal and shear stress components from the x, y to the x', y' axes, can be developed in a general manner and expressed as a set of stress transformation equations.

16.2.1 Sign Convention

As shown on Figure 16.3a, the $+x$ and $+x'$ axes are used to define the outward normal on the right face of the element, so that σ_x and $\sigma_{x'}$ are positive, when they act in the positive directions of the x and x' directions, and τ_{xy} and $\tau_{x'y'}$ are positive when they act in the positive y and y' directions.

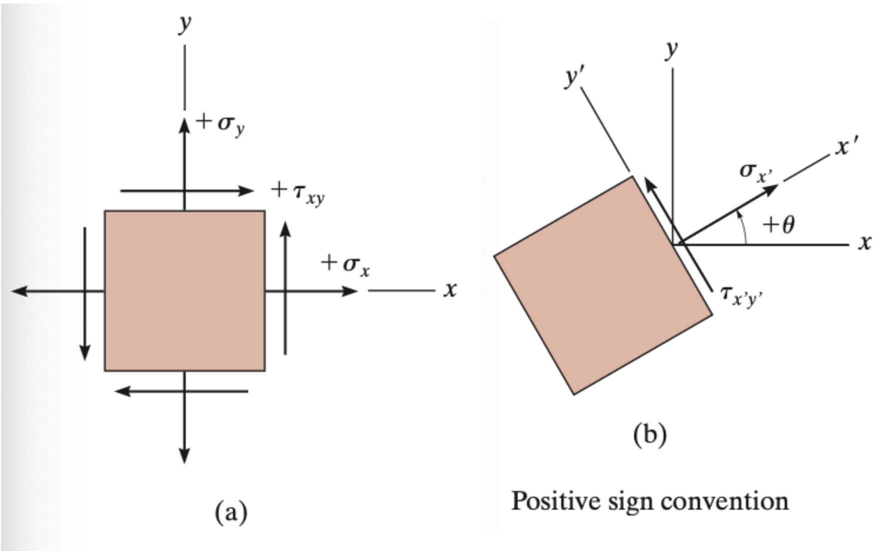

Figure 16.3: Sign Convention definition

The orientation of the face upon which the normal and shear stress components are to be determined will be defined by the angle θ which is measured from the $+x$ to the $+x'$ axis, as shown on Figure 16.3b. This angle is positive following the right hand rule.

16.2.2 Normal and Shear Stress Components

The element on Figure 16.4a is sectioned along the inclined plane and the segment shown on Figure 16.4b is isolated. Assuming the sectioned area is ΔA then the horizontal and vertical faces of the segment have an area of $\Delta A \sin\theta$ and $\Delta A \cos\theta$, respectively.

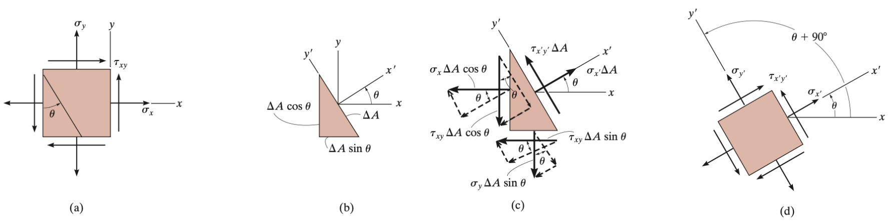

Figure 16.4: Sectioned element

The resulting free-body diagram is shown on Figure 16.4c. Using the equations of equilibrium along the x' and y' axes, we can obtain a direct solution for $\sigma_{x'}$ and $\tau_{x'y'}$ as

$$\sigma_{x'}\Delta A - (\tau_{xy}\Delta A\sin\theta)\cos\theta - (\sigma_y\Delta A\sin\theta)\sin\theta - (\tau_{xy}\Delta A\cos\theta)\sin\theta - (\sigma_x\Delta A\cos\theta)\cos\theta = 0$$
$$\sigma_x\cos^2\theta + \sigma_y\sin^2\theta + \tau_{xy}\left(2\sin\theta\cos\theta\right) = \sigma_{x'}$$
$$\tau_{x'y'}\Delta A + (\tau_{xy}\Delta A\sin\theta)\sin\theta - (\sigma_y\Delta A\sin\theta)\cos\theta - (\tau_{xy}\Delta A\cos\theta)\cos\theta + (\sigma_x\Delta A\cos\theta)\sin\theta = 0$$
$$(\sigma_y - \sigma_x)\sin\theta\cos\theta + \tau_{xy}\left(\cos^2\theta - \sin^2\theta\right) = \tau_{x'y'}.$$

These two equations can be simplified as:

$$\sigma_{x'} = \frac{\sigma_x+\sigma_y}{2} + \frac{\sigma_x-\sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$
$$\tau_{x'y'} = -\frac{\sigma_x-\sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta.$$

16.3 Principal Stresses and Maximum In-Plane Shear Stress

Since σ_x, σ_y, and τ_{xy} are all constant, then the magnitudes of $\sigma_{x'}$ and $\tau_{x'y'}$ only depend on the angle of inclination θ of the planes of which these stresses act.

16.3.1 In-Plane Principal Stresses

To determine the maximum and minimum normal stress, we differentiate the formula for normal stress with respect to θ and set the result equal to zero. Solving for $\theta = \theta_p$, which gives the planes of maximum and minimum normal stress yields:

$$\tan 2\theta_p = \frac{\tau_{xy}}{\frac{\sigma_x-\sigma_y}{2}}.$$

To obtain the maximum and minimum normal stress we must then substitute the found angles into the formula for the normal stress. After substituting and simplifying, the maximum and minimum stresses are:

$$\sigma_{1,2} = \frac{\sigma_x+\sigma_y}{2} + \sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + \tau_{xy}^2}.$$

These two values, with $\sigma_1 \geq \sigma_2$ are called the in-plane principal stresses and the planes on which they act are called the principal planes of stress. No shear-stress acts on the principal planes.

16.3.2 Maximum In-Plane Shear Stress

The orientation of the element subjected to maximum shear stress can be determined by taking the derivative of the formula for shear stress with respect to θ and setting the result equal to zero, yielding:

$$\tan 2\theta_s = \frac{-\frac{\sigma_x-\sigma_y}{2}}{\tau_{xy}}.$$

An element subjected to maximum shear stress will be oriented 45° from the position of an element subjected to the principal stress.

The maximum shear stress can be found as:

$$\tau_{\text{max in-plane}} = \pm\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}.$$

The average normal stress on the planes of maximum in-plane shear stress is

$$\sigma_{\text{avg}} = \frac{\sigma_x + \sigma_y}{2}.$$

Lecture 25: Mohr's circle

1. December 2025

16.4 Mohr's Circle – Plane Stresses

We can write the equations for the normal and shear stress as

$$\sigma_{x'} - \left(\frac{\sigma_x - \sigma_y}{2}\right) = \left(\frac{\sigma_x - \sigma_y}{2}\right)\cos 2\theta + \tau_{xy}\sin 2\theta$$
$$\tau_{x'y'} = -\left(\frac{\sigma_x - \sigma_y}{2}\right)\sin 2\theta + \tau_{xy}\cos 2\theta.$$

θ can now be eliminated by squaring each equation and adding them, yielding

$$\left(\sigma_{x'} - \left(\frac{\sigma_x + \sigma_y}{2}\right)\right)^2 + \tau_{x'y'}^2 = \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2.$$

Since σ_x, σ_y, and τ_{xy} are known constants, the above equation can be written in a compact form as:

$$(\sigma_{x'} - \sigma_{\text{avg}})^2 + \tau_{x'y'}^2 = R^2 \tag{8}$$

where

$$\sigma_{\text{avg}} = \frac{\sigma_x + \sigma_y}{2}$$
$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}.$$

We establish coordinate axes σ positive to the right and τ positive downward and then plot Equation (8), we see that the equation represents a circle with radius R, and center on the σ axis at point $C(\sigma_{\text{avg}}, 0)$, shown on Figure 16.5.

Each point on Mohr's circle represents the two stress components $\sigma_{x'}$ and $\tau_{x'y'}$ acting on the side of the element defined by the outward x' axis, when this axis is in a specific direction θ.

16.5 Absolute Maximum Shear Stress

The strength of a ductile material depends upon its ability to resist shear stress. Therefore, it is important to find the absolute maximum shear stress in a material subject to a loading.

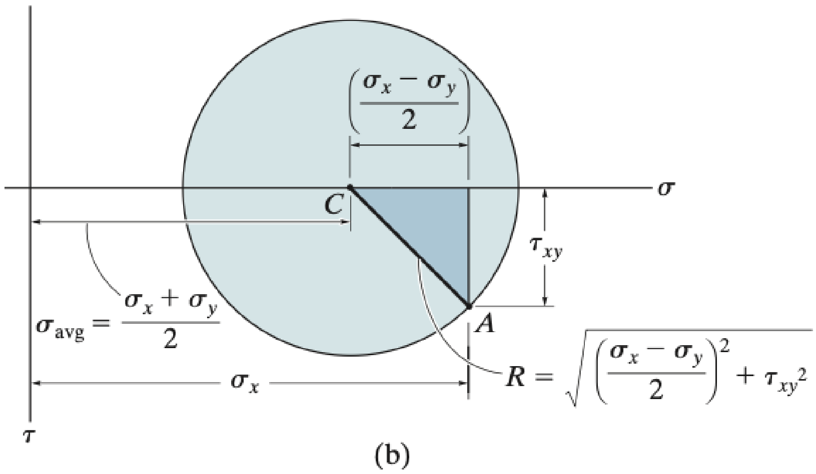

Figure 16.5: Mohr's circle.

When σ_1 and σ_2 have the same sign the maximum absolute shear stress is:

$$\tau_{\text{abs max}} = \frac{\sigma_1}{2}.$$

When σ_1 and σ_2 have opposite signs the maximum shear stress is:

$$\tau_{\text{abs max}} = \frac{\sigma_1 - \sigma_2}{2}.$$

Lecture 26: Plane strain transformation & strain rosettes — 4. December 2025

16.6 Plane Strain

The general state of strain at a point in a body is represented by a combination of three components of normal strain, ϵ_x, ϵ_y, ϵ_z and three components of shear strain γ_{xy}, γ_{xz}, γ_{yz}. The normal strains cause a change in the volume of the element, and the shear strains cause a change in shape.

16.7 General Equations of Plane-Strain Transformation

The plane-strain analysis it is important to establish strain transformation equations that can be used to determine the components of normal and shear strain at a point $\epsilon_{x'}$, $\epsilon_{y'}$, $\gamma_{x'y'}$, provided the components ϵ_x, ϵ_y, γ_{xy} are known (as depicted on Figure 16.6.

Sign Convention The normal strains ϵ_x and ϵ_y in Figure 16.6a are positive if they cause elongation along the x and y axes and the shear strain γ_{xy} is positive if the interior angle AOB becomes smaller than 90°.

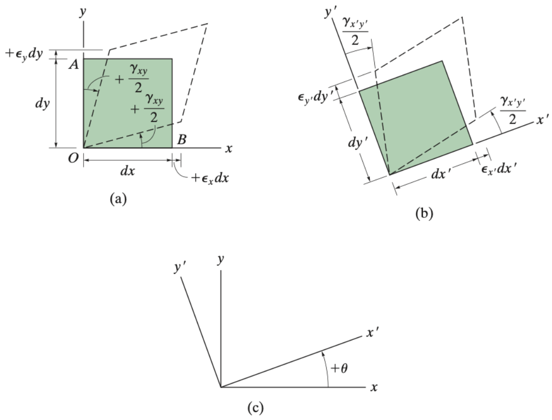

Figure 16.6: Plane-strain transformation.

Finally, if the angle between the x and x' axes is θ then θ will be positive if it follows the curl of the right-hand fingers, i.e. counterclockwise.

Normal and Shear Strains We have the general results:

$$\begin{aligned} \epsilon_{x'} &= \frac{\epsilon_x + \epsilon_y}{2} + \frac{\epsilon_x - \epsilon_y}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta \\ \epsilon_{y'} &= \frac{\epsilon_x + \epsilon_y}{2} - \frac{\epsilon_x - \epsilon_y}{2}\cos 2\theta - \frac{\gamma_{xy}}{2}\sin 2\theta \\ \frac{\gamma_{x'y'}}{2} &= -\left(\frac{\epsilon_x - \epsilon_y}{2}\right)\sin 2\theta + \frac{\gamma_{xy}}{2}\cos 2\theta. \end{aligned}$$

Principal Strains Like stress, an element can be oriented at a point so that the element's deformation is caused only by normal strains, with no shear strain. When this occurs, the normal strains are referred to as *principal strains*, and if the material is isotropic, the axes along which these strains occur will coincide with the axes of principal stress.

The direction of the x' axis and the two values of the principal strains ϵ_1 and ϵ_2 are determined from:

$$\begin{aligned} \tan 2\theta_p &= \frac{\gamma_{xy}}{\epsilon_x - \epsilon_y} \\ \epsilon_{1,2} &= \frac{\epsilon_x + \epsilon_y}{2} \pm \sqrt{\left(\frac{\epsilon_x - \epsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}. \end{aligned}$$

Maximum In-Plane Shear Strain We also have that:

$$\begin{aligned} \tan 2\theta_s &= -\left(\frac{\epsilon_x - \epsilon_y}{\gamma_{xy}}\right) \\ \frac{\gamma_{\text{max,in-plane}}}{2} &= \sqrt{\left(\frac{\epsilon_x - \epsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2} \\ \epsilon_{\text{avg}} &= \frac{\epsilon_x + \epsilon_y}{2}. \end{aligned}$$

16.8 Strain Rosettes

The normal strain on the free surface of a body can be measured in a particular direction using an electrical resistance strain gauge. Unfortunately, the shear strain cannot be measured directly with a strain gauge, so to obtain ϵ_x, ϵ_y, γ_{xy}, we must use a cluster of three strain gauges that are arranges in a pattern called a *strain rosette.*

To show how this is done, we consider the general case of arranging the gauges at angles θ_a, θ_b, θ_c as shown in Figure 16.7a. If the readings ϵ_a, ϵ_b, ϵ_c are obtained, we can determine the strain components using the normal strain transformation equations for each gauge resulting in

$$\begin{aligned}
\epsilon_a &= \epsilon_x \cos^2\theta_a + \epsilon_y \sin^2\theta_a + \gamma_{xy}\sin\theta_a\cos\theta_a \\
\epsilon_b &= \epsilon_x \cos^2\theta_b + \epsilon_y \sin^2\theta_b + \gamma_{xy}\sin\theta_b\cos\theta_b \\
\epsilon_c &= \epsilon_x \cos^2\theta_c + \epsilon_y \sin^2\theta_c + \gamma_{xy}\sin\theta_c\cos\theta_c .
\end{aligned}$$

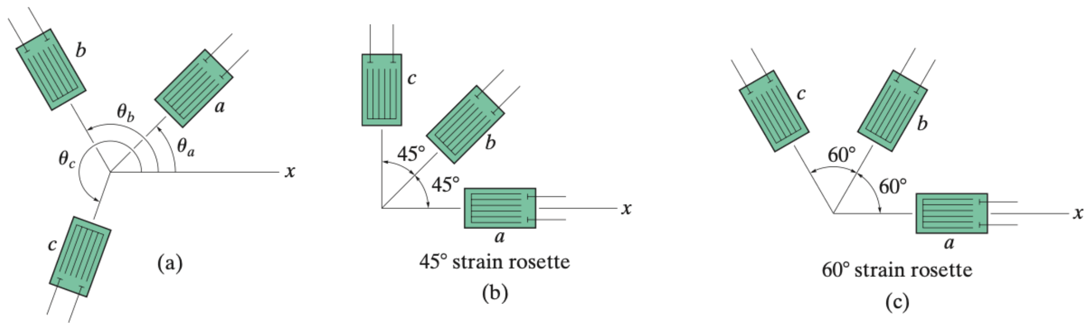

Figure 16.7: Strain Rosettes.

45° Rosette In the case of the 45° or "rectangular" strain rosette depicted on Figure 16.7b, $\theta_a = 0°$, $\theta_b = 45°$, $\theta_c = 90°$, so

$$\begin{aligned}
\epsilon_x &= \epsilon_a \\
\epsilon_y &= \epsilon_c \\
\gamma_{xy} &= 2\epsilon_b - (\epsilon_a + \epsilon_c) .
\end{aligned}$$

60° Rosette For the 60° strain rosette depicted on Figure 16.7c, $\theta_a = 0°$, $\theta_b = 60°$, $\theta_c = 120°$. Here

$$\begin{aligned}
\epsilon_x &= \epsilon_a \\
\epsilon_y &= \frac{1}{3}(2\epsilon_b + 2\epsilon_c - \epsilon_a) \\
\gamma_{xy} &= \frac{2}{\sqrt{3}}(\epsilon_b - \epsilon_c) .
\end{aligned}$$

Lecture 27: Buckling of columns 11. December 2025

17 Buckling of Columns

17.1 Critical Load

A member must not only satisfy strength and deflection requirements, but it must also be stable – This is particularly important if the member is long and slender and it supports compressive loading that becomes large enough to cause the members to suddenly deflect laterally or sideways. These members are called *columns*, and the lateral deflection that occurs is called *buckling*. Buckling can lead to sudden and dramatic failure of a mechanism, and as a result, attention must be given to the design of columns so they can safely support their intended loadings without buckling.

Figure 17.1: Column on verge of buckling.

The maximum axial load a column can support on the verge of buckling is called the *critical load*, P_{cr}, shown on Figure 17.1. Any slight additional loading beyond this will cause the column to buckle and deflect laterally.

We consider this instability as a two-bar mechanism consisting of rigid weightless bars that are pin-connected as shown on Figure 17.2a. When the bars are in the vertical position, the spring, having a stiffness k, is unstretched, and a *small* vertical force $\vec{P}$ is applied at the top of one of the bars. To upset equilibrium, the pin at A is displaced by a small amount Δ, shown on Figure 12.5b. As shown on the free-body diagram of the pin on Figure 12.5c, the sprinc will produce a restoring force $F = k\Delta$ in order to resist the two horizontal components $P_x = P\tan\theta$. Since θ is small $\Delta \approx \theta L/2$ and $\tan\theta \approx \theta$. Thus the restoring spring force becomes $F = k\theta L/2$ and the disturbing force is $2P_x = 2P\theta$.

If the restoring force is greater than the disturbing force, then, we can solve for P which gives:

$$P < \frac{kL}{4} \qquad \text{Stable equilibrium.}$$

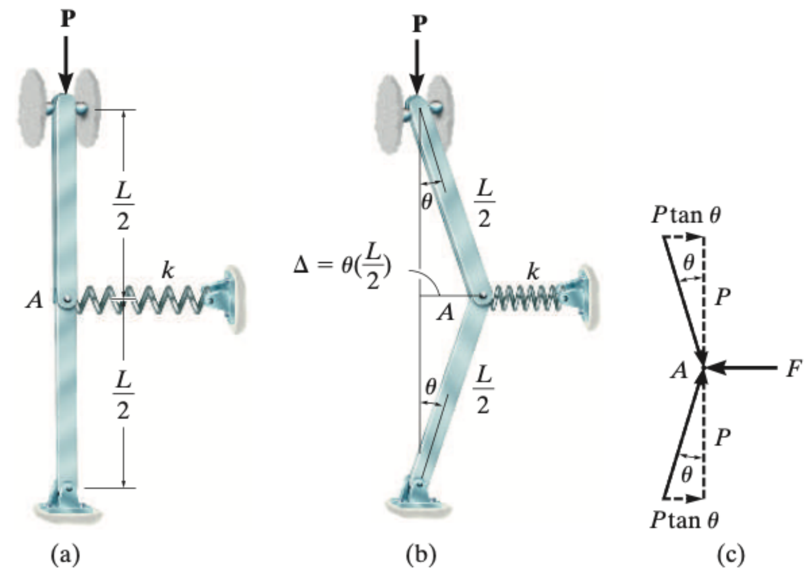

Figure 17.2: Simplified model of buckling column.

Or

$$P > \frac{kL}{4} \qquad \text{Unstable equilibrium.}$$

The critical load is right at

$$P_{\text{cr}} = \frac{kL}{4}.$$

Since P_{cr} is independent of the small displacement θ of the bars, any slight disturbance given to the mechanism will not cause it to move further out of equilibrium, nor will it be restored to its original position. Instead the bard will *remain* in the deflected position.

These three different states of equilibrium are represented on Figure 17.3. The transition point where the load is equal to the critical value $P = P_{\text{cr}}$ is called the *bifurcation point*. Here the bars will be in neutral equilibrium for any small value of θ.

17.2 Ideal Column with Pin Supports

We consider a pin and roller supported column as shown on Figure 17.4, which we will term an *ideal column*, meaning it is made of a homogeneous linear elastic material and is perfectly straight before loading. Here, the load is applied through the centroid of the section.

The tendency for a column to remain stable or become unstable when subjected to an axial load depends upon its ability to resist bending. Hence, in order to determine the critical load and the buckled shape of the column, we use

$$EI\frac{\mathrm{d}^2v}{\mathrm{d}x^2} = M.$$

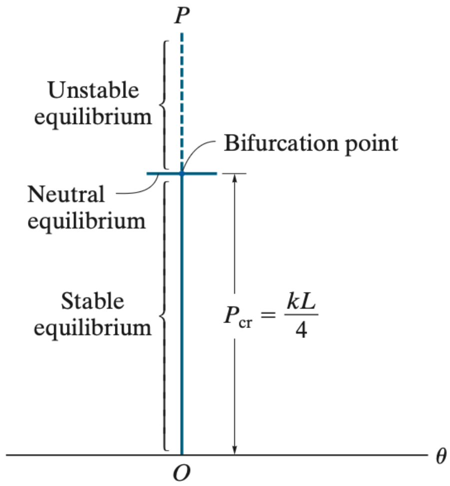

Figure 17.3: Bifurcation point.

A free-body diagram of a segment of the column in the deflected position is shown on Figure 17.5. Moment equilibrium requires

$$M = -Pv$$

so

$$EI\frac{\mathrm{d}^2v}{\mathrm{d}x^2} = -Pv$$

$$\frac{\mathrm{d}^2v}{\mathrm{d}x^2} + \left(\frac{P}{EI}\right)v = 0.$$

The general solution to this homogeneous, second-order linear differential equation with constant coefficients is

$$v = C_1 \sin\left(\sqrt{\frac{P}{EI}}x\right) + C_2 \cos\left(\sqrt{\frac{P}{EI}}x\right).$$

Since $v = 0$ at $x = 0$, $C_2 = 0$ and since $v = 0$ at $x = L$, then

$$C_1 \sin\left(\sqrt{\frac{P}{EI}}L\right) = 0.$$

This requires

$$\sin\left(\sqrt{\frac{P}{EI}}L\right) = 0$$

which is satisfied if

$$\sqrt{\frac{P}{EI}}L = n\pi$$

or

$$P = \frac{n^2\pi^2EI}{L^2}, \quad n = 1, 2, 3, \ldots.$$

The smallest value of P is for $n = 1$ (n is the number of curves in the deflected shape of the column), so the critical load is:

$$P_{\mathrm{cr}} = \frac{\pi^2EI}{L^2}$$

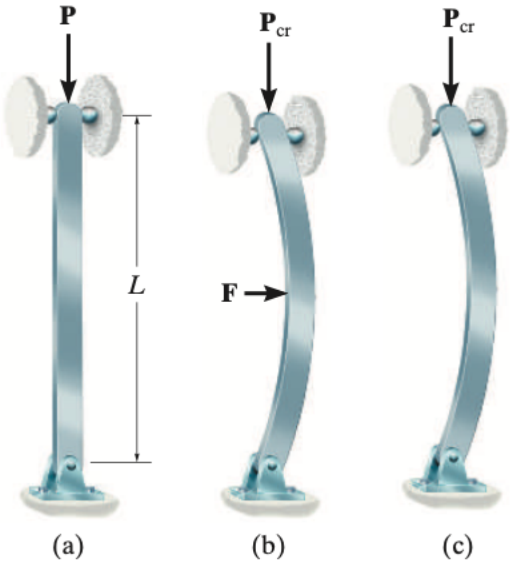

Figure 17.4: Column with pin and roller supports.

which is also referred to as the *Euler load*. The corresponding buckled shape is

$$v = C_1 \sin \frac{\pi x}{L}.$$

The constant C_1 represents the maximum deflection $v_{\max}$ which occurs at the midpoint.

I.e.

$$P_{\text{cr}} = \frac{\pi^2 EI}{L^2}$$

where

- P_{cr} is the critical or maximum axial load on the column before it begins to buckle.
- E is the modulus of elasticity
- I is the *least* moment of inertia for the column's cross-sectional area
- L is the unsupported length of the column

For design purposes, this can be written in terms of stress using $I = Ar^2$, where A is the cross-sectional area and r is the *radius of gyration* of the cross sectional area. We have,

$$\sigma_{\text{cr}} = \frac{\pi^2 E}{\left(\frac{L}{r}\right)^2}$$

Where

- σ_{cr} is the critical stress
- E is the modulus of elasticity

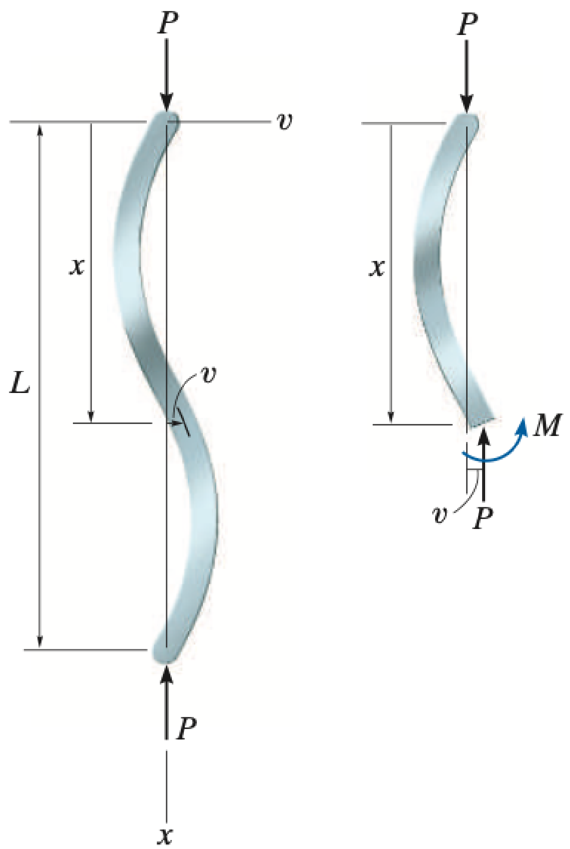

Figure 17.5: Free body diagram of buckled column.

- L is the unsupported length of the column
- r is the smallest radius of gyration of the column given by $r = \sqrt{I/a}$

17.3 Columns Having Various Types of Supports

The Euler load above was derived for column that is pin connected. Columns can, however, also be supported differently.

Effective Length To use Euler's formula, for columns having different types of supports, we will modify the column length L to represent the distance between the points of zero moment on the column. This distance is called the column's effective length L_e. For a pin-ended column $L_3 = L$ as shown on Figure 17.6a. For the fixed-end and free-ended column the effective length is $L_3 = 2L$ as shown on Figure 17.6b. Other examples are shown on Figure 17.6c and Figure 17.6d.

Rather than specifying the column's effective length, many design codes provide column formulas that employ a dimensionless coefficient K called the *effective length factor*, defined from

$$L_e = KL.$$

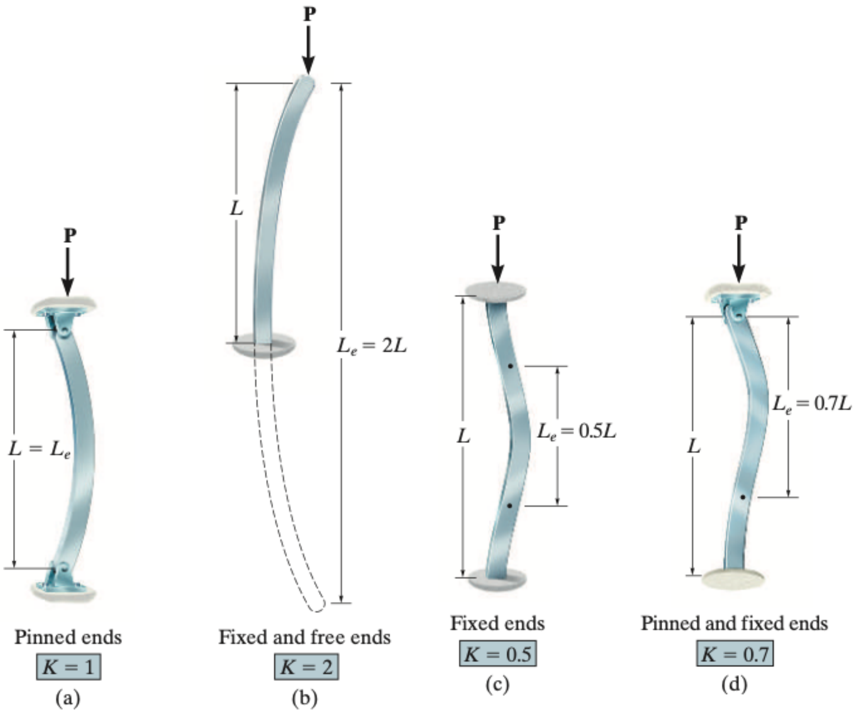

Figure 17.6: Effective lengths of columns fixed in different ways.

Based on this generality, we can write Euler's formula as

$$P_{\rm cr} = \frac{\pi^2 EI}{(KL)^2}$$

or

$$\sigma_{\rm cr} = \frac{\pi^2 E}{\left(\frac{KL}{r}\right)^2}.$$