Department of Mechanical and Production Engineering

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Lecture Notes

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Contents

Lecture 1: Introduction 26. Januar 2026

1 Introduction

This collection of notes are the personal notes of me, Noah Hansen, for the course "Theory of Elasticity" (Da: Elasticitetsteori) taught on the 4th semester of the bachelor on the mechanical engineering programme at Aarhus Universitet.

The notes will take base in the lectures by Assistant Professor Matteo Pezzulla and "Introduction to Linear Elasticity" by Phillip L. Gould and Yuan Feng.

1.1 Coordinate Rotation

Given a position vector $\boldsymbol{r}$ to a point P, we can resolve it into components with respect to two different Cartesian systems, x_i and x'_i, having a common origin.

The point P has coordinates $P(x_1, x_2, x_3) = P(x_i)$ in the unprimed system and $P(x'_1, x'_2, x'_3) = P(x'_i)$ in the primed system. The linear transformation between the coordinates is then:

$$
\begin{aligned}
x'_1 &= \alpha_{11}x_1 + \alpha_{12}x_2 + \alpha_{13}x_3 \\
x'_2 &= \alpha_{21}x_2 + \alpha_{22}x_2 + \alpha_{23}x_3 \\
x'_3 &= \alpha_{31}x_1 + \alpha_{32}x_2 + \alpha_{33}x_3
\end{aligned}
$$

or

$$x'_i = \alpha_{ij}x_j.$$

Each of the nine quantities α_{ij} is the cosine of the angle between the ith primed and the jth unprimed axis, i.e.

$$\alpha_{ij} = \cos\left(x'_i, x_j\right) = \frac{\partial x_j}{\partial x'_i} = \boldsymbol{e}'_i \cdot \boldsymbol{e}_j = \cos\left(\boldsymbol{e}'_i, \boldsymbol{e}_j\right).$$

These α_{ij}'s are known as direction cosines.

1.2 Cartesian Tensors

A tensor of order n is a set of 3^n quantities which transform from one coordinate system x_i to another x'_i, by the following law:

Table 1: Tranformation laws for Tensors

n	Order	Transformation law
0	Zero (scalar)	$A'_i = A_i$
1	One (vector)	$A'_i = \alpha_{ij}A_j$
2	Two (dyadic)	$A'_{ij} = \alpha_{ik}, \alpha_{jl}, A_{kl}$
3	Three	$A'_{ijk} = \alpha_{il}\alpha_{jm}\alpha_{kn}A_{lmn}$
4	Four	$A'_{ijkl} = \alpha_{im}\alpha_{jn}\alpha_{kp}\alpha_{lq}A_{mnpq}$

Order zero and order one tensors (scalars and vectors are already familiar quantities, wheres higher-order tensors are not. These tensors are, however, useful to describe physical quantities with a corresponding number of associated directions. Second-order tensors (dyadics) are especially prevalent for elasticity and the corresponding transformation may be carried out in matrix format as:

$$\boldsymbol{A}' = \boldsymbol{R}\boldsymbol{A}\boldsymbol{R}^T.$$

in which

$$\boldsymbol{R} = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{A}' = \begin{bmatrix} A'_{11} & A'_{12} & A'_{13} \\ A'_{21} & A'_{22} & A'_{23} \\ A'_{31} & A'_{32} & A'_{33} \end{bmatrix}.$$

1.2.1 Algebra of Cartesian Tensors

Arithmetic and algebra for tensors is similar to matrix operations when it comes to addition, subtraction, equality, and scalar multiplication. Multiplication of two tensors of order n and m produces a now tensor of order $n+m$, e.g.

$$A_i B_{jk} = C_{ijk}.$$

1.3 Operational Tensors

Additional tensor operations are made possible by the Kronecker delta δ_{ij}, defined as,

$$\delta_{ij} \equiv \begin{cases} 1 & i = j \\ 0 & i \neq j. \end{cases}$$

and the permutation tensor ϵ_{ijk} defined such that:

$$\begin{aligned} \epsilon_{ijk} &= \frac{1}{2}(i-j)(j-k)(k-i) \\ &= \begin{cases} 0 & \text{If any two of } i, j, k \text{ are equal} \\ 1 & \text{For an even permutation (forward on the number line } 1,2,3,1,2,3,\ldots) \\ -1 & \text{For an odd permutation (backward on the number line } 3,2,1,3,2,1,\ldots). \end{cases} \end{aligned}$$

The Kronecker delta δ_{ij} changes the subscripts in a tensor by multiplication:

$$\begin{aligned} \delta_{ij} A_i &= \delta_{1j} A_1 + \delta_{2j} A_2 + \delta_{3j} A_3 = A_j \\ \delta_{ij} C_{ijk} &= \delta_{11} C_{11k} + \delta_{22} C_{22k} + \delta_{33} C_{33k} = C_{iik} = A_K. \end{aligned}$$

It is also useful in vector algebra and for coordinate transformations. E.g. starting with the dot product of two unit vectors:

$$\boldsymbol{e}_i \cdot \boldsymbol{e}_j = \delta_{ij} \tag{1}$$

and seeking the component of vector $\boldsymbol{A}$ in the j-direction A_j we do:

$$\begin{aligned} \boldsymbol{e}_j \cdot \boldsymbol{A} &= \left(\boldsymbol{e}_j \cdot \boldsymbol{e}_i\right) A_i \\ &= \delta_{ji} A_i \\ &= A_j. \end{aligned}$$

The permutation tensor ϵ_{ijk} is useful for vector cross-product operations:

$$\epsilon_{ijk} A_j B_k \boldsymbol{e}_i = \boldsymbol{A} \times \boldsymbol{B}.$$

Lecture 2: Traction, Stress, and Equilibrium 2. Februar 2026

2 Traction, Stress, and Equilibrium

2.1 State of Stress

For a deformable body subject to external loading we focus on an element with area ΔA_n oriented according to the unit normal $\boldsymbol{n}$, we call the resultant force $\Delta \boldsymbol{F}_n$ and the resultant moment $\Delta \boldsymbol{M}_n$. Both are vector quanitites, and are *not*, in general, parallel to $\boldsymbol{n}$. We represent the intensity of these as

$$\lim_{\Delta A_n \to 0} \frac{\Delta \boldsymbol{F}_n}{\Delta A_n} = \frac{\mathrm{d}\boldsymbol{F}_n}{\mathrm{d}A_n} = \boldsymbol{T}_n$$
$$\lim_{\Delta A_n \to 0} \frac{\Delta \boldsymbol{M}_n}{\Delta A_n} = \frac{\mathrm{d}\boldsymbol{M}_n}{\mathrm{d}A_n} = \boldsymbol{C}_n$$

where $\boldsymbol{T}_n$ is known as the stress vector or the *traction*, and $\boldsymbol{C}_n$ is called the *couple-stress vector*.

2.2 Components of Stress

We consider an infinitesimal rectangular prism erected at the point from Section 2.1 and erect a set of Cartesian coordinates x_i parallel to the sides as shown on Figure 2.1. Each of these coordinate axis has a corresponding unit vector $\boldsymbol{e}_i$.

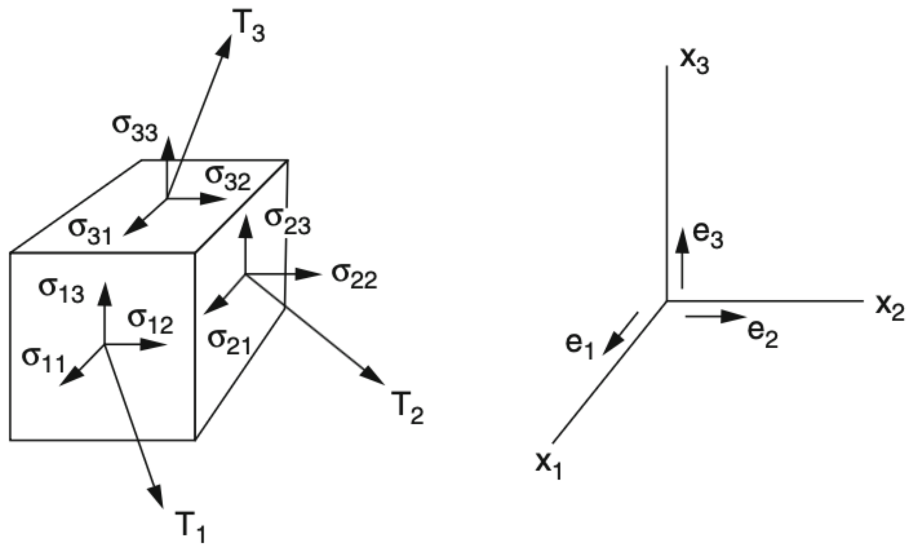


Figure 2.1: Components of stress.

Each traction may be written in terms of its Cartesian components in the form

$$\boldsymbol{T}_i = \sigma_{ij} \boldsymbol{e}_j \tag{2}$$

which is shown for $\boldsymbol{T}_1$ below

$$\boldsymbol{T}_1 = \sigma_{11}\boldsymbol{e}_1 + \sigma_{12}\boldsymbol{e}_2 + \sigma_{13}\boldsymbol{e}_3.$$

In Equation (2), the coefficients σ_{ij} are known as stresses, and the entire array of stresses forms the so-called Cauchy stress tensor. The subscript and sign convention for stress components σ_{ij} are:

1. The first subscript i refers to the *normal* $\boldsymbol{e}_i$, which denotes the face on which $\boldsymbol{T}_i$ acts.
2. The second subscript j corresponds to the *direction* $\boldsymbol{e}_j$ in which the stress acts.

3. The normal components σ_{ii} are positive if they produce tension and negative if they produce compression. The shearing components $\sigma_{ij}(i \neq j)$ are positive if directed in the positive x_j direction while acting on the face with the unit normal $\boldsymbol{e}_i$ or if directed in the negative x_j direction while acting on the face with normal $-\boldsymbol{e}_i$

2.3 Stress at a Point

It can be shown that to completely describe the state of stress at a point, it is sufficient to specify the tractions $\boldsymbol{T}_i$ on each of the three planes $\boldsymbol{e}_i$ – i.e. this is equivalent to specifying the nine stress components σ_{ij}.

$$T_i = \sigma_{ji} n_j. \tag{3}$$

And if the components T_i are known the magnitude of T_n is evaluated as

$$T_n = |\boldsymbol{T}_n| = (T_i T_i)^{\frac{1}{2}}.$$

2.3.1 Stress on a Normal Plane

It is sometimes useful to resolve $\boldsymbol{T}_n$ into components that are normal and tangential to the differential surface element $\mathrm{d}A_n$ as depicted on Figure 2.2. Here, the normal component σ_{nn} is calculated by

$$\begin{aligned}\sigma_{ns} &= |\boldsymbol{N}| = \boldsymbol{T}_n \cdot \boldsymbol{n} \\ &= T_i \boldsymbol{e}_i \cdot \boldsymbol{n} = T_i n_i.\end{aligned}$$

or using Equation (3) as

$$\sigma_{nn} = \sigma_{ji} n_j n_i. \tag{4}$$

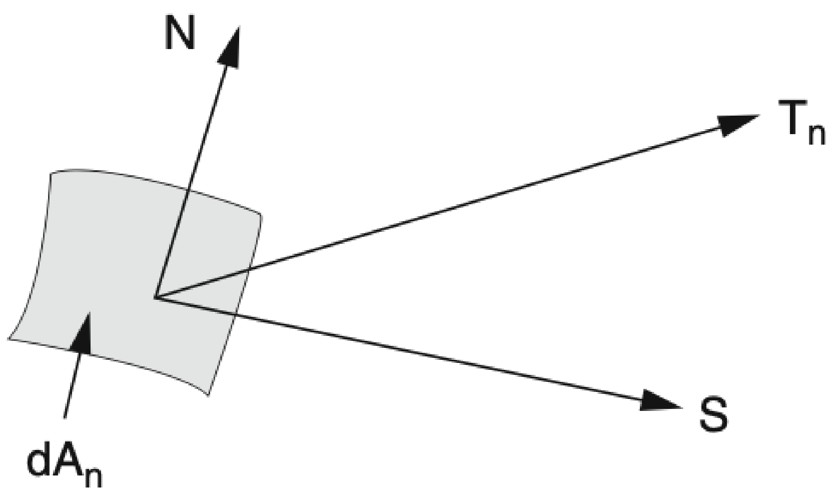


Figure 2.2: Differential surface element.

The tangential component σ_{ns} is

$$\begin{aligned}\sigma_{nn} = |\boldsymbol{S}| &= \boldsymbol{T}_n \cdot \boldsymbol{s} \\ &= T_i \boldsymbol{e}_i \cdot \boldsymbol{s} \\ &= T_i s_i \\ &= \sigma_{ji} n_j s_i.\end{aligned}$$

where

$$s_i = \boldsymbol{e}_i \boldsymbol{s}.$$

Often it is easy to calculate σ_{ns} using the Pythagorean theorem as

$$\sigma_{ns} = \sqrt{T_i T_i - \sigma_{nn}^2}.$$

One is also able to resolve N and S into Cartesian components using (k) to designate the axes as

$$\begin{aligned}\sigma_{nn(k)} = \boldsymbol{N} \cdot \boldsymbol{e}_k &= \sigma_{nn} \boldsymbol{n} \cdot \boldsymbol{e}_k \\ &= \sigma_{nn} n_k \\ &= \sigma_{ji} n_j n_i n_k, \quad k = 1,2,3.\end{aligned}$$

For σ_{ns} subtraction gives

$$\sigma_{ns(k)} = T_k - \sigma_{nn(k)} \quad k = 1,2,3$$

which may also be written as

$$\sigma_{ns(k)} = \sigma_{jk} n_j - \sigma_{ji} n_j n_i n_k = (\delta_{ki} - n_i n_k) \sigma_{ji} n_j.$$

2.3.2 Dyadic Representation of Stress

One is able to view the stress tensor as a vector-like quantity with a magnitude and associated direction(s) specified by unit vectors. The *dyadic* is such a representation. We write the stress tensor or stress dyadic as

$$\begin{aligned}\boldsymbol{\sigma} &= \sigma_{ij} \boldsymbol{e}_i \boldsymbol{e}_j \\ &= \sigma_{11} \boldsymbol{e}_1 \boldsymbol{e}_1 + \sigma_{12} \boldsymbol{e}_1 \boldsymbol{e}_2 + \sigma_{13} \boldsymbol{e}_1 \boldsymbol{e}_3 + \sigma_{21} \boldsymbol{e}_2 \boldsymbol{e}_1 + \sigma_{22} \boldsymbol{e}_2 \boldsymbol{e}_2 + \sigma_{23} \boldsymbol{e}_2 \boldsymbol{e}_3 + \sigma_{31} \boldsymbol{e}_3 \boldsymbol{e}_1 + \sigma_{32} \boldsymbol{e}_3 \boldsymbol{e}_2 + \sigma_{33} \boldsymbol{e}_3 \boldsymbol{e}_3\end{aligned}$$

wherein the double vectors are called *dyads*. The corresponding tractions are evaluated by an operation analogous to the dot product:

$$\boldsymbol{T}_i = \boldsymbol{\sigma} \cdot \boldsymbol{e}_i = \sigma_{ij} \boldsymbol{e}_j.$$

Similarly, the normal and tangential components of the traction $\boldsymbol{T}_n$ on a plane defined by normal $\boldsymbol{n}$ are

$$\begin{aligned}\sigma_{nn} &= \boldsymbol{\sigma} \cdot \boldsymbol{n} \cdot \boldsymbol{n} \\ &= \boldsymbol{T}_n \cdot \boldsymbol{n} \\ &= \sigma_{ij} n_i n_j\end{aligned}$$

and

$$\begin{aligned}\sigma_{ns} &= \boldsymbol{\sigma} \cdot \boldsymbol{n} \cdot \boldsymbol{s} \\ &= \boldsymbol{T}_n \cdot \boldsymbol{s} \\ &= \sigma_{ij} n_i s_j.\end{aligned}$$

Lecture 3: The Cauchy theorem and principal stresses 9. Februar 2026

Lecture 4: Equilibrium Equations & Principle of Angular Momentum 16. Februar 2026

2.4 Equilibrium

2.4.1 Foundations

The principle of linear momentum is

$$\int_V \boldsymbol{f}\,\mathrm{d}V + \int_A \boldsymbol{T}\,\mathrm{d}A = \int_V \rho\ddot{\boldsymbol{u}}\,\mathrm{d}V \tag{5}$$

where ρ is the mass density and $\boldsymbol{u}$ is the displacement.

The principle of angular momentum is

$$\int_V (\boldsymbol{r}\times\boldsymbol{f})\,\mathrm{d}V + \int_A (\boldsymbol{r}\times\boldsymbol{T})\,\mathrm{d}A = \int_V (\boldsymbol{r}\times\rho\ddot{\boldsymbol{u}})\,\mathrm{d}V \tag{6}$$

where $\boldsymbol{r}$ is the position vector.

We will also utilize the divergence theorem:

$$\int_V \nabla\cdot\boldsymbol{G}\,\mathrm{d}V = \int_A \boldsymbol{n}\cdot\boldsymbol{G}\,\mathrm{d}A. \tag{7}$$

These can each be written in component form using:

$$\begin{aligned}
\boldsymbol{f} &= f_i\boldsymbol{e}_i\\
\boldsymbol{T} &= \sigma_{ji}n_j\boldsymbol{e}_i\\
\boldsymbol{r} &= x_j\boldsymbol{e}_j\\
\boldsymbol{r}\times\boldsymbol{f} &= \epsilon_{ijk}x_jf_k\boldsymbol{e}_i\\
\boldsymbol{r}\times\boldsymbol{T} &= \epsilon_{ijk}x_j\sigma_{lk}n_l\boldsymbol{e}_i.
\end{aligned}$$

For static problems, the rhs of Equations (5) to (7) are zero. Substituting this and using the above component-form we get

$$\int_V f_i\,\mathrm{d}V + \int_A \sigma_{ji}n_j\,\mathrm{d}A = 0 \tag{8}$$

$$\int_V \epsilon_{ijk}x_jf_k\,\mathrm{d}V + \int_A \epsilon_{ijk}x_j\sigma_{lk}n_l\,\mathrm{d}A = 0 \tag{9}$$

$$\int_V G_{i,i}\,\mathrm{d}V = \int_A n_iG_i\,\mathrm{d}A. \tag{10}$$

2.4.2 Linear Momentum

If we apply Equation (10) with $G_i = \sigma_{ji}$ to Equation (8) we get:

$$\int_V f_i + \sigma_{ji,j}\,\mathrm{d}V = 0.$$

For all elements of V to be in equilibrium, the above equation is satisfied if the integrand vanishes, i.e.

$$\sigma_{ji,j} + f_i = 0. \tag{11}$$

This represents three equilibrium equations in terms of nine unknown components of stress σ_{ij}.

2.4.3 Angular Momentum

We apply Equation (10) with $G_i = \epsilon_{ijk} x_j \sigma_{lk}$ and $n_i = n_l$ to Equation (9) and get:

$$\int_V \epsilon_{ijk} x_j f_k + \left(\epsilon_{ijk} x_j \sigma_{lk}\right)_{,l} \, \mathrm{d}V = 0. \tag{12}$$

The second term is expanded as

$$\left(\epsilon_{ijk} x_j \sigma_{lk}\right)_{,l} = \epsilon_{ijk} \sigma_{lk} x_{j,l} + \epsilon_{ijk} x_j \sigma_{lk,l}.$$

Also we use that x_j only changes in the j'th direction: i.e.

$$x_{j,l} = \delta_{jl}.$$

Hence, Equation (12) becomes:

$$\int_V \epsilon_{ijk} \left(x_j f_k + \delta_{jl} \sigma_{lk} + x_j \sigma_{lk,l}\right) \mathrm{d}V = 0$$

the first and third term combines to

$$\epsilon_{ijk} x_j \left(f_k + \sigma_{lk,l}\right) = 0$$

from Equation (11). The remaining term becomes

$$\epsilon_{ijk} \delta_{jl} \sigma_{lk} = \epsilon_{ijk} \sigma_{jk}.$$

So the equation reduces to

$$\int_V \epsilon_{ijk} \sigma_{jk} \, \mathrm{d}V = 0$$

which holds for

$$\epsilon_{ijk} \sigma_{jk} = 0.$$

In general, this is satisfied for

$$\sigma_{ij} = \sigma_{ji}.$$

This allows us to rewrite the component definition of traction as

$$T_i = \sigma_{ij} n_j$$

and Equation (11) as

$$\sigma_{ij,j} + f_i = 0.$$

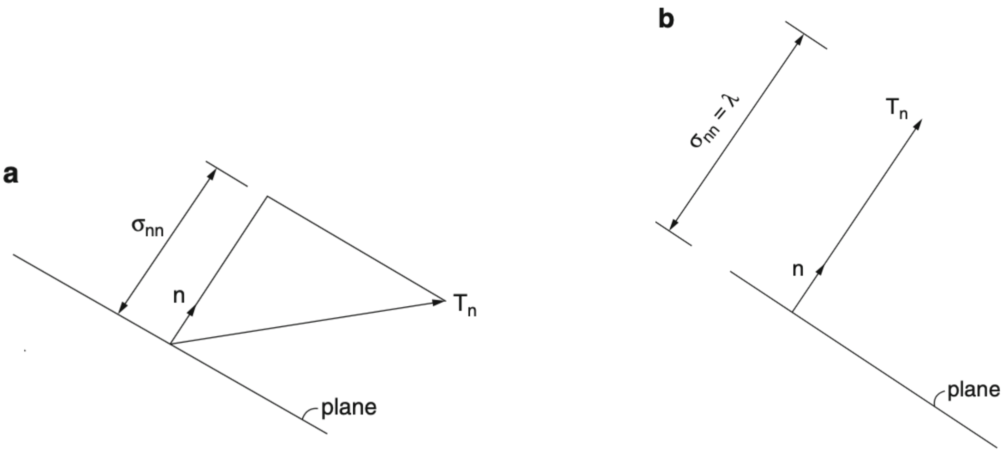
Which corresponds to the statically indeterminate system:

$$\begin{aligned} \sigma_{11,1} + \sigma_{12,2} + \sigma_{13,3} + f_1 &= 0 \\ \sigma_{21,1} + \sigma_{22,2} + \sigma_{23,3} + f_2 &= 0 \\ \sigma_{31,1} + \sigma_{32,2} + \sigma_{33,3} + f_3 &= 0. \end{aligned}$$

2.5 Principal Stress

2.5.1 Definition and Derivation

We consider the normal section through a body shown on Figure 2.3a. The traction $\boldsymbol{T}_n$ acts on the plane defined by $\boldsymbol{n}$ that appears edgewise.



(a) Normal section (b) Principal plane

Figure 2.3: Normal section and principal plane

Using the dyadic form,

$$\sigma_{nn} = \boldsymbol{T}_n \cdot \boldsymbol{n} = \boldsymbol{\sigma} \cdot \boldsymbol{n} \cdot \boldsymbol{n}$$

we seek the orientation of a plane, given by $\boldsymbol{n}$, such that σ_{nn} is an extremum. Such a plane is called a *principal* plane.

Hence,

$$\begin{aligned}\frac{\mathrm{d}\sigma_{nn}}{\mathrm{d}n} &= \boldsymbol{n} \cdot \frac{\mathrm{d}(\boldsymbol{\sigma} \cdot \boldsymbol{n})}{\mathrm{d}n} + (\boldsymbol{\sigma} \cdot \boldsymbol{n}) \cdot \frac{\mathrm{d}\boldsymbol{n}}{\mathrm{d}n} \\ &= 2(\boldsymbol{\sigma} \cdot \boldsymbol{n}) \cdot \frac{\mathrm{d}\boldsymbol{n}}{\mathrm{d}tn} \\ &= 2\boldsymbol{T}_n \cdot \frac{\mathrm{d}\boldsymbol{n}}{\mathrm{d}n} \\ &= 0.\end{aligned}$$

For $\boldsymbol{T}_n \cdot \frac{\mathrm{d}\boldsymbol{n}}{\mathrm{d}n}$ to equal 0, $\boldsymbol{T}_n$ must be normal to $\frac{\mathrm{d}\boldsymbol{n}}{\mathrm{d}n}$. Furthermore, $\frac{\mathrm{d}\boldsymbol{n}}{\mathrm{d}n}$ is normal to $\boldsymbol{n}$, so $\boldsymbol{T}_n$ must be parallel to $\boldsymbol{n}$. This condition corresponding to the extremum is shown in Figure 2.3b. Here, the tangential component $\sigma_{ns} = 0$ and the normal component for this case is designated as

$$\sigma_{nn} = |\boldsymbol{T}_n| = \lambda$$

which may also be represented as

$$\boldsymbol{T}_n = \lambda \boldsymbol{n}$$

or in component form as

$$T_i = \lambda n_i.$$

Also, we know that

$$T_i = \sigma_{ji} n_j.$$

Equating these gives

$$\lambda n_i = \lambda n_j \delta_{ij} = \sigma_{ji} n_j$$

or

$$\left(\sigma_{ji} - \lambda \delta_{ij}\right) n_j = 0. \tag{13}$$

This is a set of three equations in four unknowns, n_1, n_2, n_3, and λ. The components n_j determine the orientation of the principal plane and λ is the principal stress. This is a linear eigenvalue problem, and with Cramer's rule, it can be shown that a nontrivial solution may be found if the determinant of the coefficients of n_j vanishes, i.e.

$$\left|\sigma_{ji} - \lambda\delta_{ij}\right| = 0$$

or

$$\begin{vmatrix} \sigma_{11} - \lambda & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} - \lambda & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} - \lambda \end{vmatrix} = 0.$$

This expands into a cubic equation for λ indicating that three principal stresses, $\lambda_1 = \sigma^{(1)}, \lambda_2 = \sigma^{(2)}, \lambda_3 = \sigma^{(3)}$, exist. For each principal stress, a corresponding distinct principal orientation may be evaluated by substituting, a known value of $\sigma^{(k)}$ into Equation (13). As this still represents three equations in four unknowns, the additional supplementary relationship:

$$n_j^{(k)} n_j^{(k)} = 1$$

expressing the "length" of the unit normal for $k = 1, 2, 3$ must also be introduced.

2.6 Properties and Special States of Stress

2.6.1 Plane Stress

Consider a plane passing through P with no stresses, i.e. $\boldsymbol{T} = 0$. The projection theorem states that the traction on any other plane passing through P must be perpendicular to the normal to the stress-free plane; hence, it is parallel to that plane. Consequently, if the traction on a plane is parallel to another plane, then the latter plane will be stress free. This is known as a state of *plane stress* and is of practical use since this allows us to reduce the dimensionality of elasticity problems.

If one of the principal stresses is zero, the state of stress is plane.

2.6.2 Linear Stress

For two nonparallel planes on which $\boldsymbol{T} = 0$, the traction on any other plane must be parallel to both of these planes and also to their line of intersection. This is a state of *linear stress* and is essentially a one-dimensional problem, where two principal stresses vanish.

2.6.3 Pure Stress

If, for any coordinate system

$$\sigma_{11} = \sigma_{22} = \sigma_{33} = 0$$

the state of stress is called *pure shear* since only shear stresses, $\sigma_{ij}, i \neq j$ exist. For pure shear, $\sigma_{ii} = 0$ for any coordinate system.

2.6.4 Hydrostatic Stress

If the principal stresses $\sigma^{(1)}$ are all equal, then the stress state is hydrostatic, and the shearing stresses are zero.

Lecture 5: The Strain Tensor 23. Februar 2026

3 Deformations

3.1 Strain

Consider the undeformed and deformed positions of an elastic body shown on Figure 3.1. It is convenient to designate two sets of Cartesian coordinates x_i and X_i with $i = 1,2,3$, called *initial* coordinates and *final* coordinates, respectively. Now, we designate a reference point $P(\overline{x}_i)$ on the undeformed body and $p(\overline{X}_i)$ on the deformed body, where $\overline{x}_i$ and $\overline{X}_i$ indicate specific values of the coordinates x_i and X_i. Also we locate the neighbouring points $Q(\overline{x}_i + \mathrm{d}\overline{x}_i)$ and $Q(\overline{X}_i + \mathrm{d}\overline{X}_i)$, separated from P and p by $\mathrm{d}s$ and $\mathrm{d}S$ respectively. The latter are given in component form by

$$
\begin{aligned}
(\mathrm{d}s)^2 &= \mathrm{d}\overline{x}_1^2 + \mathrm{d}\overline{x}_2^2 + \mathrm{d}\overline{x}_3^2 = \mathrm{d}\overline{x}_i \mathrm{d}\overline{x}_i \\
(\mathrm{d}S)^2 &= \mathrm{d}\overline{X}_1^2 + \mathrm{d}\overline{X}_2^2 + \mathrm{d}\overline{X}_3^2 = \mathrm{d}\overline{X}_i \mathrm{d}\overline{X}_i .
\end{aligned}
$$

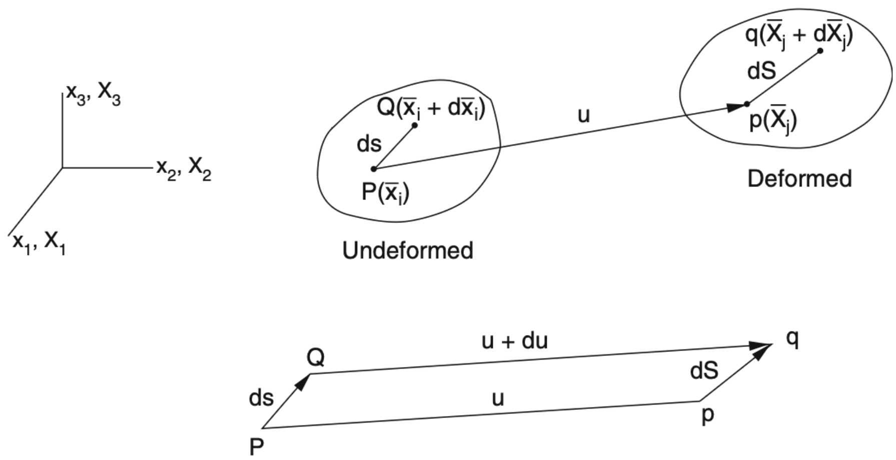


Figure 3.1: Displacement vector

In Figure 3.1, $\boldsymbol{u}$ represents the displacement of P to p and $\boldsymbol{u} + \mathrm{d}\boldsymbol{u}$ is the displacement of Q to q. From the vector diagram we note

$$
\mathrm{d}\boldsymbol{s} + (\boldsymbol{u} + \mathrm{d}\boldsymbol{u}) = \boldsymbol{u} + \mathrm{d}\boldsymbol{S} \implies \mathrm{d}\boldsymbol{u} = \mathrm{d}\boldsymbol{S} - \mathrm{d}\boldsymbol{s}.
$$

We consider the relationship in component form:

$$
(\mathrm{d}S)^2 - (\mathrm{d}s)^2 = \mathrm{d}\overline{X}_i \, \mathrm{d}\overline{X}_i - \mathrm{d}\overline{x}_i \, \mathrm{d}\overline{x}_i .
$$

Now, we only consider the initial coordinates x_i in this next step, so that

$$
\begin{aligned}
\overline{X}_i &= \overline{X}_i(x_1, x_2, x_3) \\
\mathrm{d}\overline{X}_i &= \frac{\partial X_i}{\partial x_1}\mathrm{d}\overline{x}_1 + \frac{\partial X_i}{\partial x_2}\mathrm{d}\overline{x}_2 + \frac{\partial X_i}{\partial x_3}\mathrm{d}\overline{x}_3 \\
&= X_{i,j}\, \mathrm{d}\overline{x}_j \\
\boldsymbol{u} &= u_1 \boldsymbol{e}_{x_1} + u_2 \boldsymbol{e}_{x_2} + u_3 \boldsymbol{e}_{x_3} = u_i \boldsymbol{e}_{x_i} \\
u_i &= X_i - x_i .
\end{aligned}
$$

Applying this to the component form relationship:

$$
\begin{aligned}
(\mathrm{d}S)^2-(\mathrm{d}s)^2 &= X_{i,j}\,\mathrm{d}\overline{x}_j\,X_{i,k}\,\mathrm{d}\overline{x}_k-\mathrm{d}\overline{x}_i\,\mathrm{d}\overline{x}_i\\
&=\left(X_{i,j}X_{i,k}-\delta_{ij}\delta_{ik}\right)\mathrm{d}\overline{x}_j\,\mathrm{d}\overline{x}_k\\
&=\left((x_i+u_i)_{,j}\,(x_i+u_i)_{,k}-\delta_{jk}\right)\mathrm{d}\overline{x}_j\,\mathrm{d}\overline{x}_k\\
&=\left(\left(\delta_{ij}+u_{i,j}\right)\left(\delta_{ik}+u_{i,k}\right)-\delta_{ik}\right)\mathrm{d}\overline{x}_j\,\mathrm{d}\overline{x}_k\\
&=\left(\delta_{jk}+u_{j,k}+u_{k,j}+u_{i,j}u_{i,k}-\delta_{jk}\right)\mathrm{d}\overline{x}_j\,\mathrm{d}\overline{x}_k\\
&=\left(u_{j,k}+u_{k,j}+u_{i,j}u_{i,k}\right)\mathrm{d}\overline{x}_j\,\mathrm{d}\overline{x}_k\\
&=2\epsilon^L_{ij}\,\mathrm{d}\overline{x}_i\,\mathrm{d}\overline{x}_j
\end{aligned}
$$

in which

$$
\epsilon^L_{ij}=\frac{1}{2}\left(u_{i,j}+u_{j,i}+u_{k,i}u_{k,j}\right)
$$

are components of the Lagrangian strain tensor $\boldsymbol{\epsilon}^L$.

Doing the same for the final coordinates X_i as the basis we obtain

$$
\begin{aligned}
(\mathrm{d}S)^2-(\mathrm{d}s)^2 &= \left(u_{i,j}+u_{j,i}-u_{k,i}u_{k,j}\right)\mathrm{d}\overline{X}_i\,\mathrm{d}\overline{X}_j\\
&=2\epsilon^E_{ij}\,\mathrm{d}\overline{X}_i\,\mathrm{d}\overline{X}_j
\end{aligned}
$$

where

$$
\epsilon^E_{ij}=\frac{1}{2}\left(u_{i,j}+u_{j,i}-u_{k,i}u_{k,j}\right)
$$

are components of the *Eulerian* strain tensor $\boldsymbol{\epsilon}^E$.

If the displacement gradients $u_{k,i}$ are small, then their products are so too and may be dropped. For such cases:

$$
\epsilon^L_{ij}=\epsilon^E_{ij}=\epsilon_{ij}=\frac{1}{2}\left(u_{i,j}+u_{j,i}\right).
$$

The components ϵ_{ij} refer to the small strain tensor $\boldsymbol{\epsilon}$.

3.2 Physical Interpretation of Strain Tensor

Consider the components $\epsilon_{(ii)}$ called *extensional* strains. E.g. a fiber parallel to x_2 with initial length $\mathrm{d}s$ and final length $\mathrm{d}S$, giving $\mathrm{d}x_1=\mathrm{d}x_2=0$ and $\mathrm{d}x_2=\mathrm{d}s$. Hence,

$$
(\mathrm{d}S)^2-(\mathrm{d}s)^2=2\epsilon_{22}\,\mathrm{d}s^2
$$

for the case of infinitesimal strain.

This gives:

$$
\begin{aligned}
\frac{\mathrm{d}S-\mathrm{d}s}{\mathrm{d}s}&=\frac{2\epsilon_{22}\,\mathrm{d}s^2}{\mathrm{d}s\,(\mathrm{d}S+\mathrm{d}s)}\\
&=\frac{2\epsilon_{22}\,\mathrm{d}s^2}{2\,\mathrm{d}s^2}\\
&=\epsilon_{22}\\
\implies \epsilon_{22}&=\frac{\mathrm{d}S-\mathrm{d}s}{\mathrm{d}s}
\end{aligned}
$$

which is the relative elongation of the fiber.

The components ϵ_{ij} are called *shearing strains* and may be interpreted by considering intersecting fibers initially parallel to two of the coordinate axes. E.g. the x_1- and x_3-axes as shown in Figure 3.2. These fibers have initial lengths $\mathrm{d}s^{(1)}$ and $\mathrm{d}s^{(3)}$, final length $\mathrm{d}S^{(1)}$ and $\mathrm{d}S^{(3)}$ and have rotated through angles $\beta^{(1)}$ and $\beta^{(3)}$, respectively. The total change in the initially right angle is

$$\beta^{(1)} + \beta^{(3)} = \beta.$$

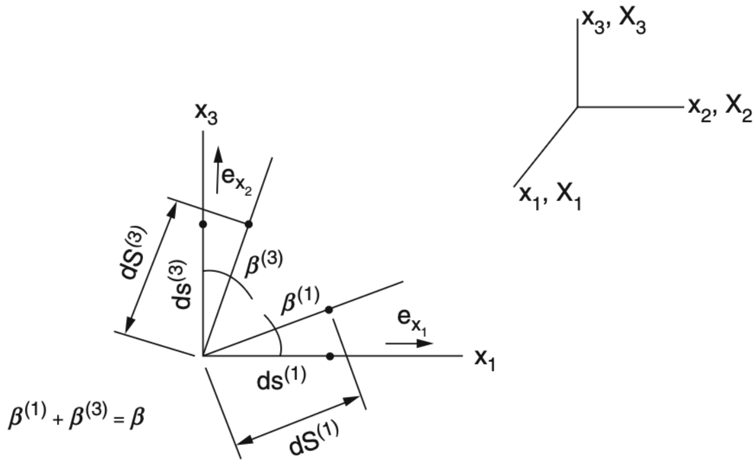


Figure 3.2: Shearing strains

Vectors coinciding with the fibers can be written in component form as:

$$\mathrm{d}\boldsymbol{S}^{(1)} = \mathrm{d}S_i^{(1)} \boldsymbol{e}_{x_i}$$

and

$$\mathrm{d}\boldsymbol{S}^{(3)} = \mathrm{d}S_i^{(3)} \boldsymbol{e}_{x_i}.$$

Using these, we can show that

$$\beta = (u_{1,3} + u_{3,1}) = 2\epsilon_{13}.$$

I.e. the strain component ϵ_{13} is one half the increase in the angle between two fibers initially parallel to the x_1- and x_3-axes.

Since the strain tensor is formed from the displacement gradients, it is interesting to separate $u_{i,j}$ into two parts:

$$u_{i,j} = \epsilon_{ij} + \omega_{ij}$$

where

$$\omega_{ij} = \frac{1}{2}\left(u_{i,j} - u_{j,i}\right).$$

For $i = j, \omega_{ij} = 0$, while for $i \neq j$, $\omega_{ij} = -\omega_{ji}$; thus, the ω_{ij} are elements of a skew-symmetric tensor $\boldsymbol{\omega}$. Using the example from before,

$$\omega_{13} = \frac{1}{2}\left(u_{1,3} - u_{3,1}\right).$$

Also

$$\beta = \beta^{(1)} + \beta^{(3)} = (u_{1,3} + u_{3,1}) = 2\epsilon_{13}.$$

By comparison we get

$$\omega_{13} = \frac{\beta^{(1)} - \beta^{(3)}}{2}$$

which can be interpreted as the average rotation of all fibers around the x_2 axis. For a rigid body rotation, $\epsilon_{13} = 0, \beta^{(1)} = -\beta^{(3)}$, and $\omega_{13} = \beta^{(1)}$. Hence, $\boldsymbol{\omega}$ is called the rotation tensor.

Lecture 6: Physical interpretation of the strain tensor

2. Marts 2026

3.3 Principal Strain

As the strains have the same second-order tensor character as the stresses, it is expected that we are able to identify a system of principal strains that are the eigenvalues, determined from the characteristic equation

$$\left|\epsilon_{ji} - \lambda\delta_{ji}\right| = 0$$

which is analogous to the result for stresses. The three roots λ_1, λ_2, and λ_3 are the principal strains $\epsilon^{(k)}$ with $k = 1, 2$, and 3.

The corresponding eigenvectors designate the directions associated with each of the principal strains and are computed from

$$\left(\epsilon_{ji} - \lambda\delta_{ji}\right) n_j = 0.$$

The directions of principal strain are denoted by $n_j^{(k)}$, with $k = 1, 2$, and 3. These directions are mutually perpendicular and, for isotropic materials, conincide with the directions of the principal stresses. Which makes logical sense, as the largest stress is also expected to cause the largest strain.

3.4 Volume and Shape Change

Sometimes it is convenient to separate the components of strain into those that cause changes in volume and those that cause change in the shape of a differential element.

Considering a volume element oriented with the principal directions and with shearing stresses equal to zero. The original length of each side is $\mathrm{d}x^{(k)}$ and the final length as $\mathrm{d}X^{(k)}$ where

$$\mathrm{d}X^{(k)} = \mathrm{d}x^{(k)}\left(1 + \epsilon^{(k)}\right).$$

The original volume is found as

$$\mathrm{d}v = \prod_{k=1}^{3} \mathrm{d}x^{(k)}$$

and the final volume as

$$\begin{aligned}
\mathrm{d}V &= \prod_{k=1}^{3} \mathrm{d}X^{(k)} \\
&= \prod_{k=1}^{3}\left(1 + \epsilon^{(k)}\right) \mathrm{d}x^{(k)} \\
&= \mathrm{d}v \prod_{k=1}^{3}\left(1 + \epsilon^{(k)}\right).
\end{aligned}$$

We define the *dilation* as the relative change in volume

$$\begin{aligned}
\Delta &= \frac{\mathrm{d}V - \mathrm{d}v}{\mathrm{d}v} \\
&= \frac{\mathrm{d}v\left(\left(\prod_{k=1}^{3}\left(1+\epsilon^{(k)}\right)\right)-1\right)}{\mathrm{d}v} \\
&= \left(\left(1+\epsilon^{(1)}\right)\left(1+\epsilon^{(2)}\right)\left(1+\epsilon^{(3)}\right)\right)-1 \\
&\approx \epsilon^{(1)}+\epsilon^{(2)}+\epsilon^{(3)} \\
&= \prod_{k=1}^{3}\epsilon^{(k)}
\end{aligned}$$

which is the sum of the diagonal terms of the principal strain tensor.

The dilation Δ is also equal to the sum of the diagonal strains ϵ_{ii}, and the divergence of the displacement vector $\nabla\cdot\boldsymbol{u} = u_{i,j}$.

Changes in shape of the volume element are called *detrusions* and are denoted by shearing strains $\epsilon_{ij}, i \neq j$.

The separation of the strains into dilational and detrusional components can be formalized by resolving the strain tensor into two terms:

$$\boldsymbol{\epsilon} = \boldsymbol{\epsilon}_M + \boldsymbol{\epsilon}_D$$

where

$$\boldsymbol{\epsilon}_M = \begin{bmatrix} \epsilon_M & 0 & 0 \\ 0 & \epsilon_M & 0 \\ 0 & 0 & \epsilon_M \end{bmatrix} = \text{mean normal strain}$$

$$\boldsymbol{\epsilon}_D = \begin{bmatrix} \epsilon_{11}-\epsilon_M & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{12} & \epsilon_{22}-\epsilon_M & \epsilon_{23} \\ \epsilon_{13} & \epsilon_{23} & \epsilon_{33}-\epsilon_M \end{bmatrix} = \text{strain deviator}$$

and where

$$\begin{aligned}
\epsilon_M &= \frac{1}{3}\left(\epsilon_{11}+\epsilon_{22}+\epsilon_{33}\right) \\
&= \frac{1}{3}\left(\epsilon^{(1)}+\epsilon^{(2)}+\epsilon^{(3)}\right) \\
&= \frac{1}{3}R_1.
\end{aligned}$$

The mean normal strain corresponds to a state of equal elongation in all directions for an element at a given point. Only subject to this, the element would maintain original shape but change in volume with dilation ϵ_{ii}. Hence, $\boldsymbol{\epsilon}_M$ is the *volumetric* component of strain.

The strain deviator, or *deviatoric* component, $\boldsymbol{\epsilon}_D$, has a dilation of

$$\begin{aligned}
\Delta &= \prod_{i=1}^{3}\epsilon_D^{(i)} \\
&= \epsilon_{D_{ii}} \\
&= \left(\epsilon_{11}-\epsilon_M\right)+\left(\epsilon_{22}-\epsilon_M\right)+\left(\epsilon_{33}-\epsilon_M\right) \\
&= \epsilon_{11}+\epsilon_{22}+\epsilon_{33}-3\left(\frac{1}{3}\left(\epsilon_{11}+\epsilon_{22}+\epsilon_{33}\right)\right) \\
&= 0.
\end{aligned}$$

This corresponds to a change in shape with no change in volume of an element.

Corresponding to this, we can also define the principal stresses associated with changes in the volume and shape as

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_M + \boldsymbol{\sigma}_D$$

defined similarly as the stress, where

$$\boldsymbol{\sigma}_M = \begin{bmatrix} \sigma_M & 0 & 0 \\ 0 & \sigma_M & 0 \\ 0 & 0 & \sigma_M \end{bmatrix} = \text{mean normal stress}$$
$$\boldsymbol{\sigma}_D = \begin{bmatrix} \sigma_{11} - \sigma_M & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} - \sigma_M & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} - \sigma_M \end{bmatrix} = \text{stress deviator}$$

and where

$$\begin{aligned} \sigma_M &= \frac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33}) \\ &= \frac{1}{3}\left(\sigma^{(1)} + \sigma^{(2)} + \sigma^{(3)}\right). \end{aligned}$$

In indicial form, the volumetric and deviatoric components of stress are, respectively,

$$\begin{aligned} \sigma_{M_{ij}} &= \frac{1}{3}\sigma_{kk}\delta_{ij} \\ \sigma_{D_{ij}} &= \sigma_{ij} - \sigma_{M_{ij}} \\ &= \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij}. \end{aligned}$$

For principal axes, the stress deviator becomes

$$\begin{aligned} \boldsymbol{\sigma}_D &= \begin{bmatrix} \sigma^{(1)} - \sigma_M & 0 & 0 \\ 0 & \sigma^{(2)} - \sigma_M & 0 \\ 0 & 0 & \sigma^{(3)} - \sigma_M \end{bmatrix} \\ &= \begin{bmatrix} \sigma_D^{(1)} & 0 & 0 \\ 0 & \sigma_D^{(2)} & 0 \\ 0 & 0 & \sigma_D^{(3)} \end{bmatrix}. \end{aligned}$$

Worth noting here is that the sum of the diagonals of $\boldsymbol{\sigma}_D$ is zero as

$$\sigma_{D_{ii}} = \sigma^{(1)} + \sigma^{(2)} + \sigma^{(3)} - 3 \cdot \frac{1}{3}\left(\sigma^{(1)} + \sigma^{(2)} + \sigma^{(3)}\right) = 0.$$

Lecture 7: Volume and shape change

9. Marts 2026

3.5 Compatibility

Using $\epsilon_{ij} = 1/2\left(u_{i,j} + u_{j,i}\right)$, the components of strain may be easily computed once displacements are known provided they are once differentiable. However, the inverse problem of calculating displacements from strains is not as straight forward as there are six independent equations with only three unknowns. Conditions of compatibility imposed on the components of strain ϵ_{ij}, are both necessary and sufficient to ensure a continuous single-valued displacement field $\boldsymbol{u}$.

The standard procedure is to eliminate the displacements between the equations repeatedly to produce equations with only strains as unknowns. First we take two normal components $\epsilon_{(ll)}$ and $\epsilon_{(mm)}$ and the shearing component $\epsilon_{(lm)}$ with the summation convention suspended.

Taking second partial derivatives, we have

$$\epsilon_{(ll,mm)} = u_{(l,lmm)}$$
$$\epsilon_{(mm,ll)} = u_{(m,mll)}$$
$$.$$

$$\begin{aligned}\epsilon_{(lm,lm)} &= \frac{1}{2}\left(u_{(l,mlm)} + u_{(m,llm)}\right)\\ &= \frac{1}{2}\left(u_{(l,lmm)} + u_{(m,mll)}\right).\end{aligned}$$

Next we equate these to get

$$\epsilon_{(ll,mm)} + \epsilon_{(mm,ll)} = 2\epsilon_{(lm,lm)}.$$

Now, taking one normal component $\epsilon_{(ll)}$ and three shearing components ϵ_{lm},ϵ_{ln}, and ϵ_{nn} and write the second partials

$$\epsilon_{(ll,mm)} = u_{(l,lmn}$$
$$\epsilon_{(lm,ln)} = \frac{1}{2}\left(u_{(l,mln)} + u_{(m,lln)}\right)$$
$$\epsilon_{(ln,lm)} = \frac{1}{2}\left(u_{(l,nlm)} + u_{(n,llm)}\right)\epsilon_{(mn,ll)} = \frac{1}{2}\left(u_{(m,nll)} + u_{(n,mll)}\right).$$

These can be equated to give

$$\epsilon_{(lm,ln)} + \epsilon_{(ln,lm)} - \epsilon_{(mn,ll)} = \epsilon_{(ll,mn)}.$$

These are six equations relating the components of strain that ensure that they will integrate into a unique displacement field. These can be written explicitly as:

$$\epsilon_{11,22} + \epsilon_{22,11} = 2\epsilon_{12,12}$$
$$\epsilon_{22,33} + \epsilon_{33,22} = 2\epsilon_{23,23}$$
$$\epsilon_{33,11} + \epsilon_{11,33} = 2\epsilon_{31,31}$$
$$\epsilon_{12,13} + \epsilon_{13,12} - \epsilon_{23,11} = \epsilon_{11,23}$$
$$\epsilon_{23,21} + \epsilon_{21,23} - \epsilon_{31,22} = \epsilon_{22,31}$$
$$\epsilon_{31,32} + \epsilon_{32,31} - \epsilon_{12,33} = \epsilon_{33,12}.$$

These are known as the *St. Venant* compatibility equations and may also be extracted from the tensor equation

$$\epsilon_{ij,kl} + \epsilon_{kl,ij} = \epsilon_{ik,jk} + \epsilon_{jl,ik}$$

which represents $3^4 = 81$ equations, where only six are distinct.

Lecture 8: Linear isotropic materials and the uniaxial stress 16. Marts 2026

4 Material Behavior

4.1 Uniaxial Behavior

The most elementary and well-known description of material behavior is *Hooke's law*:

$$\sigma_{11} = E\epsilon_{11}$$

where E is the modulus of elasticity of Young's modulus. This also gives rise to stress strain curves as the one shown on Figure 4.1.

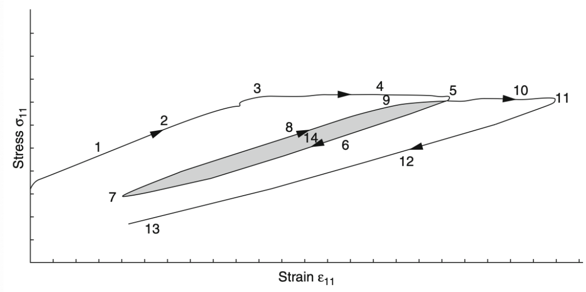


Figure 4.1: Stress-strain curve for two cycles of loading and unloading for mild steel.

Branch (1) is the initial elastic response, from which E may be calculated as the slope. In the vicinity of (2), there is a decrease in the slope, commencing at the *proportional limit* and progressing until the *yield point* is reached at (3). Yielding progresses along (4) until unloading commences at (5) and continues along (6). The unloading ceases and reloading begins at (7) to initiate another cycle that extends through (13) when the test is terminated. Region (14) is known as a *hysteresis loop* and is a measure of the energy dissipated through one excursion beyond the yield point.

4.2 Generalized Hooke's Law

We will now introduce the *strain energy density* W that is a homogeneous quadratic function of the strain components $W(\epsilon_{ij})$. For a body that is slightly strained by gradual application of the loading while the temperature remains constant, a stress component will be produced derivable as

$$\sigma_{ij} = \frac{\partial W}{\partial \epsilon_{ij}}.$$

Here, it is convenient to write W in terms of the engineering strains $\bar{\epsilon}_{ij}$. Thus

$$\sigma_{ij} = \frac{\partial W\left(\bar{\epsilon}_{ij}\right)}{\partial \bar{\epsilon}_{ij}}$$

where $\bar{\epsilon}_{ij} = \left(2 - \delta_{ij}\right)\epsilon_{ij}$.

The function should have coefficients such that $W \geq 0$ in order to ensure the stability of the body, with $W(0) = 0$ corresponding to a *natural of zero energy state*.

Now, the generalized Hooke's law is written in the form of a fourth-order tensor

$$\sigma_{ij} = E_{ijkl}\epsilon_{kl}$$

in which the 81 coefficients E_{ijkl} are called the elastic constants. Since W is continuous the order of differentiation does not matter and E_{ijkl} is symmetric. Thus the number of independent equations is reduced from nine to six, the elastic constants from 81 to 36, and then further to $((36-6)/2)+6=21$, when only half of the off-diagonal constants are counted. We represent this relationship in matrix form in terms of the engineering strains as

$$\boldsymbol{\sigma} = \boldsymbol{E}\bar{\boldsymbol{\epsilon}}$$

where

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{Bmatrix} = \begin{bmatrix} E_{1111} & E_{1122} & E_{1133} & E_{1112} & E_{1123} & E_{1131} \\ & E_{2222} & E_{2233} & E_{2212} & E_{2223} & E_{2231} \\ & & E_{3333} & E_{3312} & E_{3323} & E_{3331} \\ & & & E_{1212} & E_{1223} & E_{1231} \\ & & & & E_{2323} & E_{2331} \\ & & & & & E_{3131} \end{bmatrix} \begin{Bmatrix} \bar{\epsilon}_{11} \\ \bar{\epsilon}_{22} \\ \bar{\epsilon}_{33} \\ \bar{\epsilon}_{12} \\ \bar{\epsilon}_{23} \\ \bar{\epsilon}_{31} \end{Bmatrix}.$$

Which can also be written as

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ & & C_{33} & C_{34} & C_{35} & C_{36} \\ & & & C_{44} & C_{45} & C_{46} \\ & & & & C_{55} & C_{56} \\ & & & & & C_{66} \end{bmatrix} \begin{Bmatrix} \bar{\epsilon}_{11} \\ \bar{\epsilon}_{22} \\ \bar{\epsilon}_{33} \\ \bar{\epsilon}_{12} \\ \bar{\epsilon}_{23} \\ \bar{\epsilon}_{31} \end{Bmatrix}$$

or

$$\boldsymbol{\sigma} = \boldsymbol{C}\bar{\boldsymbol{\epsilon}}.$$

The corresponding strain energy function is

$$\begin{aligned} 2W =& C_{11}\bar{\epsilon}_{11}^2 + 2C_{12}\bar{\epsilon}_{11}\bar{\epsilon}_{22} + 2C_{13}\bar{\epsilon}_{11}\bar{\epsilon}_{33} + 2C_{14}\bar{\epsilon}_{11}\bar{\epsilon}_{12} + 2C_{15}\bar{\epsilon}_{11}\bar{\epsilon}_{23} \\ &+ 2C_{16}\bar{\epsilon}_{11}\bar{\epsilon}_{31} + C_{22}\bar{\epsilon}_{22}^2 + 2C_{23}\bar{\epsilon}_{22}\bar{\epsilon}_{33} + 2C_{24}\bar{\epsilon}_{22}\bar{\epsilon}_{12} + 2C_{25}\bar{\epsilon}_{22}\bar{\epsilon}_{23} \\ &+ 2C_{26}\bar{\epsilon}_{22}\bar{\epsilon}_{31} + C_{33}\bar{\epsilon}_{33}^2 + 2C_{34}\bar{\epsilon}_{33} + \bar{\epsilon}_{12} + 2C_{35}\bar{\epsilon}_{33}\bar{\epsilon}_{23} + 2C_{36}\bar{\epsilon}_{33}\bar{\epsilon}_{31} \\ &+ C_{44}\bar{\epsilon}_{12}^2 + 2C_{45}\bar{\epsilon}_{12}\bar{\epsilon}_{23} + 2C_{46}\bar{\epsilon}_{12}\bar{\epsilon}_{31} \\ &+ C_{55}\bar{\epsilon}_{23}^2 + 2C_{56}\bar{\epsilon}_{23}\bar{\epsilon}_{31} \\ &+ C_{66}\bar{\epsilon}_{31}^2. \end{aligned}$$

The preceding characterization is the most general with which we need to be concerned; such a material is termed *anisotropic*. Fortunately, most engineering materials possess propertied of symmetry about one or more of the planes or axes which allow the number of independent constants to be reduced.

This can be reduced for each plane of symmetry for the material ultimately leading to an isotropic material for three-axed symmetry.

The stress-strain relationship can be written in terms of the tensor components ϵ_{ij} using the Lamé constants λ and μ:

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{Bmatrix} = \begin{bmatrix} 2\mu+\lambda & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & 2\mu+\lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & 2\mu+\lambda & 0 & 0 & 0 \\ & & & 2\mu & 0 & 0 \\ & & & & 2\mu & 0 \\ & & & & & 2\mu \end{bmatrix} \begin{Bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ \epsilon_{12} \\ \epsilon_{23} \\ \epsilon_{31} \end{Bmatrix}$$

or, in indicial form,

$$\sigma_{ij} = 2\mu\epsilon_{ij} + \lambda\delta_{ij}\epsilon_{kk}.$$

This can also be inverted to express the strains in terms of the stresses:

$$\epsilon_{ij} = \frac{1}{2\mu}\sigma_{ij} - \frac{\lambda}{2\mu\left(2\mu+3\lambda\right)}\delta_{ij}\sigma_{kk}.$$

Although the Lamé constants are perfectly suitable from a mathematical standpoint, it is often more convenient to use engineering material constants that are related to measurements from elementary tests.

For a uniaxial stress with σ_{11} constant and all other $\sigma_{ij} = 0$,

$$\begin{aligned}
\sigma_{11} &= \left(2\mu+\lambda\right)\epsilon_{11} + \lambda\epsilon_{22} + \lambda\epsilon_{33},\\
0 &= \lambda\epsilon_{11} + \left(2\mu+\lambda\right)\epsilon_{22} + \lambda\epsilon_{33},\\
0 &= \lambda\epsilon_{11} + \lambda\epsilon_{22} + \left(2\mu+\lambda\right)\epsilon_{33}.
\end{aligned}$$

These can be equated to yield the basic form of Hooke's law:

$$\sigma_{11} = \frac{\mu\left(2\mu+3\lambda\right)}{\mu+\lambda}\epsilon_{11} = E\epsilon_{11}.$$

From the same uniaxial stress state, the *fractional contraction* may be computed as

$$-\frac{\epsilon_{22}}{\epsilon_{11}} = -\frac{\epsilon_{33}}{\epsilon_{11}} = \frac{\lambda}{2\left(\mu+\lambda\right)} = \nu$$

where ν is Poisson's ratio.

A third engineering constant is obtained from the state of pure shear in two dimensions, given by $\sigma_{12} = \sigma_{21} =$ constant, all other $\sigma_{ij} = 0$

$$\sigma_{12} = 2\mu\epsilon_{12} = 2G\epsilon_{12}$$

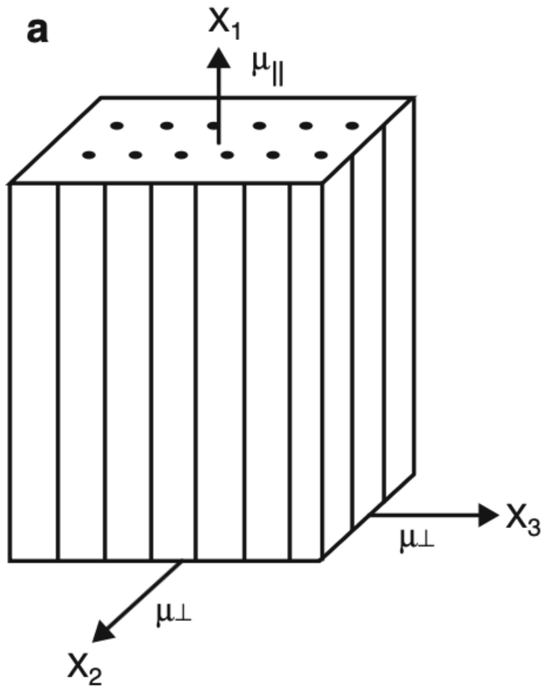
where G is the shear modulus.

Although E, ν, and G aer convenient, we realize only two of these are independent since $G = \mu$ and both E and ν are defined in terms of λ and μ. The relationships between the Lamé and engineering constants are collected as:

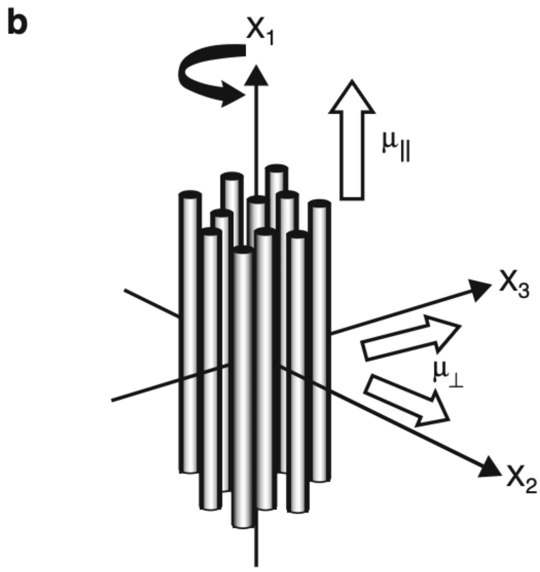
$$\begin{aligned}
\mu &= \frac{E}{2\left(1+\nu\right)}\\
\lambda &= \frac{\nu E}{\left(1+\nu\right)\left(1-2\nu\right)}\\
E &= \frac{\mu\left(2\mu+3\lambda\right)}{\mu+\lambda}\\
\nu &= \frac{\lambda}{2\left(\mu+\lambda\right)}\\
G &= \mu.
\end{aligned}$$

Another relationship between the constants is defined as the bulk modulus

$$K = \frac{E}{3\left(1+2\nu\right)}.$$



(a) Transverse isotropic material



(b) Bundled fiber configuration

Figure 4.2: Transverse isotropic material and bundled fiber configurations

Note that for λ to remain finite, Poisson's ratio must lie between

$$-1 < \nu < 0{,}5.$$

Also, for $\nu = 0{,}5$, the upper limit, the dilation $\Delta = 0$, and the material is said to be incompressible.

The explicit form of the generalized Hooke's law in terms of the engineering constants and tensoral strains is best written as

$$\epsilon = \boldsymbol{C}^{-1}\boldsymbol{\sigma}$$

or

$$\begin{Bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ \epsilon_{12} \\ \epsilon_{23} \\ \epsilon_{31} \end{Bmatrix} = \begin{bmatrix} \frac{1}{E} & -\frac{\nu}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & \frac{1}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & -\frac{\nu}{E} & \frac{1}{E} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2G} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2G} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2G} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{Bmatrix}.$$

The elements of $\boldsymbol{C}^{-1}$ are called *compliances*.

4.3 Transverse Isotropic Material

In Figure 4.2a, a configuration with the fibers oriented along the z-axis, perpendicular to the symmetric $x-y$-plane is shown. These fibers are able to stretch in a single direction as shown on Figure 4.2b. This might correspond to the muscle fibers inside a muscle.

This is a transverse isotropic material with five elastic constants, $\lambda_\perp, \lambda_\|, \lambda_M, \mu_\perp$, and $\mu_\|$. The λ terms are related to the propagation of longitudinal waves in the different directions, while the μ terms are shear moduli. The expanded stress-strain matrix $\boldsymbol{C}_{TI}$ is:

$$\boldsymbol{C}_{TI} = \begin{bmatrix} 2\mu_\parallel + \lambda_\parallel & \lambda_M & \lambda_M & 0 & 0 & 0 \\ \lambda_M & 2\mu_\perp + \lambda_\perp & \lambda_\perp & 0 & 0 & 0 \\ \lambda_M & \lambda_\perp & 2\mu_\perp + \lambda_\perp & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu_\perp & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu_\parallel & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu_\parallel \end{bmatrix}.$$

Note, that this is expressed in terms of Lamé constants μ and λ. An alternate form using engineering constants is

$$\begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ 2\epsilon_{23} \\ 2\epsilon_{13} \\ \epsilon_{12} \end{bmatrix} = \begin{bmatrix} \frac{1}{E_t} & -\frac{\nu_{pt}}{E_p} & -\frac{\nu_{pt}}{E_p} & 0 & 0 & 0 \\ -\frac{\nu_{tp}}{E_t} & \frac{1}{E_p} & -\frac{\nu_p}{E_p} & 0 & 0 & 0 \\ -\frac{\nu_{tp}}{E_t} & -\frac{\nu_p}{E_p} & \frac{1}{E_p} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\mu_p} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\mu_t} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\mu_t} \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix}.$$

The seven variables in the above compliance matrix are related by the additional equations

$$\frac{\nu_{pt}}{E_p} = \frac{\nu_{tp}}{E_t} \iff \frac{E_t}{E_p} = \frac{\nu_{tp}}{\nu_{pt}}$$
$$\mu_p = \frac{E_p}{2\left(1+\nu_p\right)}.$$

Also, as the compliance matrix should be positive definite, the following five relationships exist for stability

$$\begin{aligned} E_t, E_p, u_t, u_p &> 0 \\ \left|\nu_p\right| &< 1 \\ \nu_{pt} &< \sqrt{\frac{E_p}{E_t}} \\ \nu_{tp} &< \sqrt{\frac{E_t}{E_p}} \\ 1 - \nu_p^2 - 2\nu_{pt}\nu_{tp} - 2\nu_{pt}\nu_{tp}\nu_p &> 0. \end{aligned}$$

Lecture 9: Thermal strains

23. Marts 2026

4.4 Thermal Strains

Temperature changes may be a source of stresses in elastic systems. It is reasonable to assume that a linear relationship exists between the temperature difference from the datum value and the corresponding strain

$$\epsilon_{(ii)}(T) = \alpha\,(T - T_0)$$

in which $\epsilon_{(ii)}(T)$ is the linear strain due to the temperature difference $T - T_0$, T_0 is the datum temperature and α is the coefficient of thermal expansion.

Using this, we can write

$$\epsilon_{ij} = \frac{1}{E}\left((1+\nu)\,\sigma_{ij} - \nu\delta_{ij}\sigma_{kk}\right) + \alpha\delta_{ij}\,(T - T_0)$$

and

$$\sigma_{ij} = 2\mu\epsilon_{ij} + \lambda\delta_{ij}\epsilon_{kk} - \alpha\delta_{ij}\left(3\lambda + 2\mu\right)(T - T_0).$$

Lecture 10: The De Saint Venant Problem, Pt. I 13. April 2026

5 The De Saint Venant Problem

5.0.1 The Saint-Venant Assumptions

We consider a straight cylinder whose axis is in the $\boldsymbol{e}_3$-direction and with a cross section in the $\boldsymbol{e}_1\boldsymbol{e}_2$-plane as shown on Figure 5.1.

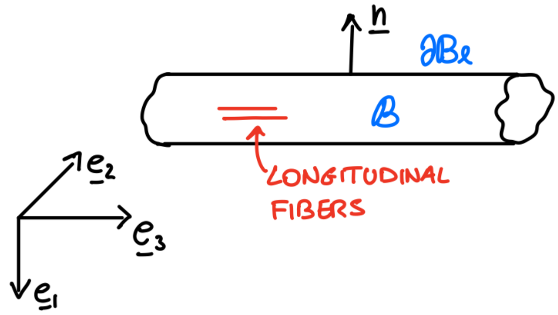


Figure 5.1: Straight cylinder

The boundary of this cylinder is split into:

$$\partial B = \partial B_l \cup \partial B_e$$

where, ∂B_l is the lateral surface and ∂B_e is the end/base surface. The outward normal on the lateral surface satisfies

$$\boldsymbol{n} \cdot \boldsymbol{e}_3 = 0$$

because the lateral surface normal has no component in the beam axis direction.

The assumptions are:

1. No loads on the lateral surface
$$\sigma\boldsymbol{n} = 0, \qquad \text{on } \partial B_l.$$
So the sides of the beam are traction-free.
2. Loads only act on the end faces. This means external loading is applied only at the bases/cross-sections.
3. Longitudinal fibers only exchange tangential stresses. I.e. if we imagine the cylinder as a bundle of fibres running along $\boldsymbol{e}_3$, then between neighboring fibres, the model only allows tangential interaction, not normal compression between fibers.

Mathematically, for every normal vector $\boldsymbol{n}$ lying in the $\boldsymbol{e}_1\boldsymbol{e}_2$-plane,

$$\boldsymbol{n} \cdot \sigma\boldsymbol{n} = 0, \qquad \forall \boldsymbol{n} : \boldsymbol{n} \cdot \boldsymbol{e}_3 = 0.$$

So if

$$\boldsymbol{n} = \begin{pmatrix} n_1 \\ n_2 \\ 0 \end{pmatrix}$$

then

$$\boldsymbol{n}\cdot\sigma\boldsymbol{n} = \sigma_{11}n_1^2 + 2\sigma_{12}n_1n_2 + \sigma_{22}n_2^2.$$

For this to be zero for all choices of n_1, n_2, every coefficient must vanish:

$$\sigma_{11} = \sigma_{22} = \sigma_{12} = 0.$$

So the stress tensor loses all the purely in-plane stress components.

5.0.2 Simplified Cauchy Stress Tensor

Because of the assumptions mentioned above, the Cauchy stress tensor reduces to:

$$\sigma = \begin{bmatrix} 0 & 0 & \sigma_{31} \\ 0 & 0 & \sigma_{31} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}.$$

The remaining stresses are, σ_{31} and σ_{32}, which are shear stresses and σ_{33} which is the normal stress in the beam direction.

So the stress state is much simpler now. The beam can carry axial normal stress, σ_{33} and shear stresses σ_{31} and σ_{32}, but not in-plane stresses like $\sigma_{11}, \sigma_{22}, \sigma_{12}$.

For a plane with normal vector

$$\boldsymbol{n} = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}$$

the traction vector acting on the plane is

$$\boldsymbol{T}_n = \sigma\boldsymbol{n}.$$

Using the simplified stress tensor, this can be written as:

$$\boldsymbol{T}_n = \begin{pmatrix} \sigma_{31}n_3 \\ \sigma_{32}n_3 \\ \sigma_{31}n_1 + \sigma_{32}n_e + \sigma_{33}n_e \end{pmatrix} = \begin{pmatrix} \sigma_{31}n_3 \\ n_{32}n_3 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \sigma_{31}n_1 + \sigma_{32}n_2 + \sigma_{33}n_3 \end{pmatrix}.$$

In the above, the first part is related to shear in the cross section and the second part acts in the longitudinal direction $\boldsymbol{e}_3$.

An important special case is a cut perpendicular to the cylinder acis,

$$\boldsymbol{n} = \boldsymbol{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

so

$$\boldsymbol{T}_{e_3} = \sigma\boldsymbol{e}_3 = \begin{pmatrix} \sigma_{31} \\ \sigma_{32} \\ \sigma_{33} \end{pmatrix}.$$

I.e. the traction on a cross section is made up of the shear stress components σ_{31}, σ_{32} and the axial/normal stress component σ_{33}.

5.1 Equilibrium Equations

Without body forces, the local equilibrium equation is

$$\operatorname{div}\sigma = \mathbf{0}$$

which in index notation is

$$\sigma_{ij,j} = 0.$$

Using the simplified stress tensor, this becomes the following three equations:

$$\begin{aligned}
\sigma_{31,3} &= 0 \\
\sigma_{32,3} &= 0 \\
\sigma_{31,1} + \sigma_{32,2} + \sigma_{33,3} &= 0.
\end{aligned}$$

I.e. the shear stresses σ_{31} and σ_{32} do not vary along the length of the cylinder. By defining the shear vector on the cross section as:

$$\boldsymbol{\tau}_3 = \begin{pmatrix} \sigma_{31} \\ \sigma_{32} \end{pmatrix}$$

the equations can be rewritten as:

$$\begin{aligned}
\boldsymbol{\tau}_{3,3} &= \mathbf{0} \\
\operatorname{div}\boldsymbol{\tau}_3 &= -\sigma_{33,3}.
\end{aligned}$$

5.1.1 Boundary Condition on the Lateral Surface

On the lateral surface,

$$\boldsymbol{n} \cdot \boldsymbol{e}_3 = 0,$$

so

$$\boldsymbol{n} = \begin{pmatrix} n_1 \\ n_2 \\ 0 \end{pmatrix}.$$

The traction is then:

$$\sigma\boldsymbol{n} = \begin{bmatrix} 0 & 0 & \sigma_{31} \\ 0 & 0 & \sigma_{32} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{pmatrix} n_1 \\ n_2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \sigma_{31}n_1 + \sigma_{32}n_2 \end{pmatrix}.$$

For the lateral surface to be traction free we have that:

$$\sigma\boldsymbol{n} = \mathbf{0}.$$

Therefore,

$$\sigma_{31}n_1 + \sigma_{32}n_2 = 0$$

now, using $\boldsymbol{\tau}_3$, this becomes

$$\boldsymbol{\tau}_3 \cdot \boldsymbol{n} = 0.$$

So the shear stress vector $\boldsymbol{\tau}_3$ must be tangent to the boundary of the cross-section.

5.1.2 Resultant Force on a Cross-Section

Now we move form local stresses to global force resultants.

On a cross section S, with normal $\boldsymbol{e}_3$, the traction vector is

$$\boldsymbol{T}_{\boldsymbol{e}_3} = \begin{pmatrix} \sigma_{31} \\ \sigma_{32} \\ \sigma_{33} \end{pmatrix}.$$

The resultant force is the integral of traction over the cross section:

$$\boldsymbol{R} = \int_S \boldsymbol{T}_{\boldsymbol{e}_3} \, \mathrm{d}S.$$

So

$$\boldsymbol{R} = \int_S \begin{pmatrix} \sigma_{31} \\ \sigma_{32} \\ \sigma_{33} \end{pmatrix} \mathrm{d}S = \begin{pmatrix} \int_S \sigma_{31} \, \mathrm{d}S \\ \int_S \sigma_{32} \, \mathrm{d}S \\ \int_S \sigma_{33} \, \mathrm{d}S \end{pmatrix}.$$

We can label this as:

$$\boldsymbol{R} = \begin{pmatrix} T_1 \\ T_2 \\ N \end{pmatrix},$$

where T_1 is the shear force in the e_1 direction, T_2 is the shear force in the e_2 direction and N is the axial normal force.

5.1.3 Resultant Moment on a Cross-Section

The moment resultant about the barycenter G of the cross-section is

$$\boldsymbol{M} = \int_S (\boldsymbol{x} - \boldsymbol{x}_G) \times \boldsymbol{T}_{\boldsymbol{e}_3} \, \mathrm{d}S.$$

The vector

$$\boldsymbol{x} - \boldsymbol{x}_G$$

is the position vector from the centroid of the cross-section to the point where the stress acts.

In the cross-section,

$$\boldsymbol{x} - \boldsymbol{x}_G = \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix}$$

the traction is

$$\boldsymbol{T}_{\boldsymbol{e}_3} = \begin{pmatrix} \sigma_{31} \\ \sigma_{32} \\ \sigma_{33} \end{pmatrix}.$$

So the cross-product is

$$(\boldsymbol{x} - \boldsymbol{x}_G) \times \boldsymbol{T}_{\boldsymbol{e}_3} = \begin{pmatrix} x_2 \sigma_{33} \\ -x_1 \sigma_{33} \\ x_1 \sigma_{32} - x_2 \sigma_{31} \end{pmatrix}.$$

We can also write this as:

$$(\boldsymbol{x} - \boldsymbol{x}_G) \times \boldsymbol{T}_{\boldsymbol{e}_3} = \begin{pmatrix} \sigma_{33} y \\ -\sigma_{33} x \\ x_1 \sigma_{32} - x_2 \sigma_{31} \end{pmatrix}.$$

The interpretation is:

$$M_1 = \int_S x_2 \sigma_{33} \, \mathrm{d}S$$

is a bending moment about e_1,

$$M_2 = \int_S -x_1\sigma_{33}\,\mathrm{d}S$$

is a bending moment about e_2, and

$$M_3 = \int_S (x_1\sigma_{32} - x_2\sigma_{31})\,\mathrm{d}S$$

is the torsional moment about e_3.

5.2 The Saint-Venant Principle

The Saint-Venant principle states that "statically equivalent force systems acting on the base surfaces of a straight cylinder give rise to the same stress/strain states, except for small regions near the base surfaces."

This means, that if two different load distributions have the same resultant force and moment, than far enough away from the loaded ends, the body "does not care" about the exact details of how the load was applied.

Lecture 11: The De Saint Venant Problem, Pt. II

20. April 2026

5.3 Finding the Barycenter of a Cross-Section

The barycenter G, also called the *centroid* for a homogeneous cross-section, is the "geometric center" of the area.

I.e. if we define a local reference frame $\{O;\boldsymbol{e}_1,\boldsymbol{e}_2\}$ in the plane of the cross-section Then every point in the cross-section will have a position vector:

$$\boldsymbol{x} = x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2.$$

The barycenter should then be:

$$\boldsymbol{x}_G = \frac{1}{A}\int_A \boldsymbol{x}\,\mathrm{d}A.$$

Or in component form as:

$$x_{G1} = \frac{1}{A}\int_A x_1\,\mathrm{d}A$$
$$x_{G2} = \frac{1}{A}\int_A x_2\,\mathrm{d}A.$$

5.4 Simple Bending: Only One Bending Moment Acts

We now consider the special case, where $M_2 \neq 0$, is the only applied moment. I.e. the body is only being bent about the e_2-axis. Here, shear forces are zero, torsional moments are zero and hence only normal stress remains.

So,

$$\boldsymbol{\tau}_3 = \boldsymbol{0}, \qquad \sigma_{33} \neq 0.$$

That means the traction on the cross-section is purely axial:

$$\boldsymbol{T}_{e_3} = \begin{pmatrix} 0 \\ 0 \\ \sigma_{33} \end{pmatrix}.$$

For Saint-Venant bending, compatibility leads to the fact that axial stress must be linear in the cross-section coordinates:

$$\sigma_{33} = a + bx_1 + cx_2.$$

Physically, this means the beam bends in such a way that one side is stretched and the other side is compressed. The axial strain varies linearly across the section, so the axial stress also varies linearly.

Because $M_1 = 0$, the stress cannot vary with x_2, so the stress depends only on x_1:

$$\sigma_{33} = \alpha x_1.$$

There is no constant term here because there is no axial force N. If there were a constant term, integrating over the area would give a nonzero axial resultant force.

So for pure bending about e_2:

$$N = 0, \qquad M_1 = 0, \qquad M_2 \geq 0.$$

Therefore σ_{33} must be linear in x_1 only.

From the previous section, the moment resultant about e_2 was

$$M_2 = -\int_A x_1 \sigma_{33} \, \mathrm{d}A.$$

Inserting $\sigma_{33} = \alpha x_1$ yields

$$M_2 = -\int_A x_1 (\alpha x_1) \, \mathrm{d}A = -\alpha \int_A x_1^2 \, \mathrm{d}A.$$

The integral

$$I_2 = \int_A x_1^2 \, \mathrm{d}A$$

is the second moment of area about e_2. So

$$M_2 = -\alpha I_2.$$

Therefore:

$$\alpha = -\frac{M_2}{I_2}.$$

Substituting back into the stress expression gives:

$$\sigma_{33}(x_1) = -\frac{M_2}{I_2} x_1.$$

5.4.1 General Case by Superposition

Because the problem is linear, we can combine the simpler solutions. If we have the axial force N and the bending moments M_1 and M_2, then the axial stress is the sum of these three contributions.

A pure axial force gives uniform stress:

$$\sigma_{33}^{(N)} = \frac{N}{A}.$$

Here, every fibre carries the same normal stress. From above, bending about e_2 results in

$$\sigma_{33}^{(M_2)} = -\frac{M_2}{I_2} x_1.$$

And similarly, bending about e_1 gives:

$$\sigma_{33}^{(M_1)} = \frac{M_1}{I_1} x_2.$$

So the total stress is:

$$\sigma_{33}(x_1, x_2) = \frac{N}{A} - \frac{M_2}{I_2} x_1 + \frac{M_1}{I_1} x_2.$$

Here, N/A shifts the whole stress distribution uniformly up or down. It is pure tension or compression.

The term $-M_2/I_2 x_1$, creates a linear stress variation in x_1 caused by bending in about e_2 and similarly for $M_1/I_1 x_2$.

Lecture 12: The De Saint Venant Problem, Pt. III

27. April 2026

5.5 Torsion of Circular Sections

We consider a straight cylinder loaded by a torsional moment about its longitudinal axis, M_3 as shown on Figure 5.2.

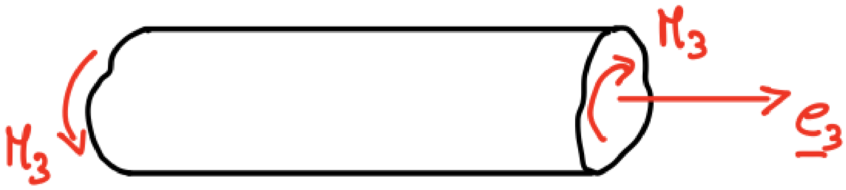


Figure 5.2: Torsionally-loaded straight bar

We restrict our attention to circular cross sections, which are either solid circular shaft, hollow circular shafts, and thin circular tubes. Circular sections are special because they do not warp under torsion, i.e. the cross-section remains a plane rigid disk under rotation.

Each cross-section rotates by an angle $\theta(x_3)$ around e_3. The twist rate is

$$\theta' = \frac{\mathrm{d}\theta}{\mathrm{d}x_3}.$$

If the torsion is uniform, θ' is constant, so

$$\theta(x_3) = \theta' x_3.$$

A point in the cross-section has position vector:

$$\boldsymbol{r}(x_1, x_2) = \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix}.$$

For a small rotation $\theta(x_3)$ around e_3, the displacement is

$$\boldsymbol{u}(x_1, x_2, x_3) = \theta(x_3)\boldsymbol{e}_3 \times \boldsymbol{r}(x_1, x_2).$$

The cross product is then:

$$\boldsymbol{e}_3 \times \boldsymbol{r} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} = \begin{pmatrix} -x_2 \\ x_1 \\ 0 \end{pmatrix}.$$

So

$$\boldsymbol{u} = \theta' x_3 \begin{pmatrix} -x_2 \\ x_1 \\ 0 \end{pmatrix}.$$

Hence,

$$
\begin{aligned}
u_1 &= -\theta' x_3 x_2 \\
u_2 &= \theta' x_3 x_1 \\
u_3 &= 0.
\end{aligned}
$$

As $u_3 = 0$, there is no warping; points only move around the shaft axis and not up or down the shaft axis.

5.5.1 Strains Produced by Torsion

The only nonzero engineering shear strains are:

$$
\begin{aligned}
\gamma_{31} &= -\theta' x_2 \\
\gamma_{32} &= \theta' x_1.
\end{aligned}
$$

Using the linear elastic shear stress relation we obtain,

$$\tau = G\gamma$$

where G is the shear modulus. We get:

$$
\begin{aligned}
\tau_{31} &= -G\theta' x_2 \\
\tau_{32} &= G\theta' x_1.
\end{aligned}
$$

So the shear stress vector in the cross-section is:

$$\boldsymbol{\tau}_3 = \begin{pmatrix} \tau_{31} \\ \tau_{32} \\ \end{pmatrix} = G\theta' \begin{pmatrix} -x_2 \\ x_1 \\ \end{pmatrix}.$$

This means the shear stress is tangent to circles around the center, with magnitude:

$$|\boldsymbol{\tau}_3| = G\theta' r.$$

We should note that

$$\operatorname{curl} \boldsymbol{\tau}_3 = 2G\theta' \boldsymbol{e}_3.$$

This comes from the field

$$\boldsymbol{\tau}_3 = \begin{pmatrix} -G\theta' x_2 \\ G\theta' x_1 \end{pmatrix}.$$

Two two-dimensional curl is

$$\frac{\partial \tau_{32}}{\partial x_1} - \frac{\partial \tau_{31}}{\partial x_2}.$$

Substituting, we obtain:

$$\frac{\partial}{\partial x_1}\left(G\theta' x_1\right) - \frac{\partial}{\partial x_2}\left(-G\theta' x_2\right) = G\theta' + G\theta' = 2G\theta'.$$

So the shear field circulates around the center.

5.5.2 Torsional Moment for a Circular Section

The torsional moment about e_3 is

$$M_3 = \int_A (\tau_{32} x_1 - \tau_{31} x_2)\, \mathrm{d}A.$$

Substituting τ_{31} and τ_{32} we get:

$$M_3 = \int_A \left(G\theta' x_1^2 + G\theta' x_2^2\right) \mathrm{d}A.$$

So

$$M_3 = G\theta' \int_A \left(x_1^2 + x_2^2\right) \mathrm{d}A.$$

Here, the integral

$$J = \int_A r^2 \,\mathrm{d}A = \int_A \left(x_1^2 + x_2^2\right) \mathrm{d}A$$

is the polar second moment of area. Hence,

$$M_3 = GJ\theta'.$$

So the torsional rigidity is:

$$C_T = GJ = \frac{M_3}{\theta'}.$$

This is the torsional equivalent of bending rigidity EI.

5.5.3 Thin-Walled Sections and the Hydrodynamic Analogy

The torsion equations are:

$$\begin{aligned} \operatorname{div} \boldsymbol{\tau}_3 &= 0 \quad \text{in } B \\ \operatorname{curl} \boldsymbol{\tau}_3 &= 2G\theta' \quad \text{in } B \\ \boldsymbol{\tau}_3 \cdot \boldsymbol{n} &= 0 \quad \text{on } \partial B. \end{aligned}$$

The last condition means the shear stress is tangent to the boundary; i.e. it cannot flow through the wall.

We can compare this to a fluid in a rotating container:

$$\begin{aligned} \operatorname{div} \boldsymbol{v} &= 0 \\ \operatorname{curl} \boldsymbol{v} &= \omega \boldsymbol{e}_3 \\ \boldsymbol{v} \cdot \boldsymbol{n} &= 0. \end{aligned}$$

So, the shear stress field behaves mathematically like a circulating velocity field, with streamlines corresponding to stress lines.

For a thin-walled section we describe the wall using s, as our coordinate along the mid-line, $\delta(s)$ as our wall thickness, L as the total length of the mid-line, t as the tangent direction along the wall and n as a normal direction through the thickness.

The wall is thin, so

$$\delta(s) \ll L.$$

A point on the wall is describes uding

$$\begin{aligned} s &\in [0, L] \\ n &\in \left[-\frac{\delta(s)}{2}, \frac{\delta(s)}{2}\right]. \end{aligned}$$

The stress components in the local frame are then τ_{zs} which is a shear stress along the wall tangent and τ_{zn} which is a shear stress through the wall thickness.

The boundary condition on the inner and outer wall surface is:

$$\tau_{zn} = 0 \qquad \text{at } n = \pm\frac{\delta(s)}{2}.$$

I.e. no shear stress flows through the free surface of the wall.

The exact curl equation is

$$\frac{\partial \tau_{zs}}{\partial n} - \frac{\partial \tau_{zn}}{\partial s} = 2G\theta'.$$

Because the wall is thin and $\tau_{zn} = 0$ on both wall surfaces, we approximate $\tau_{zn} \approx 0$ throughout the thickness. Then the equation becomes

$$\frac{\partial \tau_{zs}}{\partial n} = 2G\theta'.$$

Integrating across the thickness yields:

$$\tau_{zs}(n) = 2G\theta' n + \tau_{zs}(0).$$

So τ_{zs} varies linearly across the thickness. But because $\delta(s) \ll L$, the variation $2G\theta' n$ is small compared to the mean tangential shear stress. Therefore, we approximate the stress by its mean value across the thickness:

$$\tau_{zs}(n) \approx \overline{\tau}_{zs}.$$

5.5.4 Shear Flow

We now introduce the shear flow

$$q = \tau_z(s)\delta(s)$$

where τ_z is the average shear stress along the wall and $\delta(s)$ is the wall thickness. So q is force per unit length along the wall.

For a thin-walled closed section, the hydrodynamic analogy tells us that the shear flow is constant around the closed section, q = constant. Therefore,

$$\tau_z(s) = \frac{q}{\delta(s)}.$$

So thinner parts of the wall carry higher shear stress.

5.5.5 First Bredt Formula

The small force on a wall element is

$$\mathrm{d}\boldsymbol{F} = \tau_z(s)\boldsymbol{t}\delta(s)\,\mathrm{d}s.$$

Since

$$q = \tau_z(s)\delta(s)$$

we can write

$$\mathrm{d}\boldsymbol{F} = q\boldsymbol{t}\,\mathrm{d}s.$$

The moment of this force about the origin is

$$\mathrm{d}\boldsymbol{M}_t = \boldsymbol{r} \times q\boldsymbol{t}\,\mathrm{d}s.$$

Integrating about the mid-line yields:

$$\boldsymbol{M}_T = \int_{\partial A_m} \boldsymbol{r} \times \boldsymbol{t} q\,\mathrm{d}s.$$

Since q is constant,

$$\boldsymbol{M}_t = q\int_{\partial A_m} \boldsymbol{r} \times \boldsymbol{t}\,\mathrm{d}s.$$

Taking the scalar component along e_3, gives

$$M_t = q\int_{\partial A_m} (\boldsymbol{r}\times\boldsymbol{t})\cdot\boldsymbol{e}_3\,\mathrm{d}s = q\int_{\partial A_m} \boldsymbol{r}\cdot\boldsymbol{n}\,\mathrm{d}s.$$

By the divergence theorem,

$$\int_{\partial A_m} \boldsymbol{r}\cdot\boldsymbol{n}\,\mathrm{d}s = \int_{A_m} \operatorname{div}\boldsymbol{r}\,\mathrm{d}A.$$

Since $\boldsymbol{r} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, we have

$$\operatorname{div}\boldsymbol{r} = \frac{\partial x_1}{\partial x_1} + \frac{\partial x_2}{\partial x_2} = 2.$$

Therefore

$$\int_{A_m} \operatorname{div}\boldsymbol{r}\,\mathrm{d}A = 2A_m.$$

So

$$M_t = 2qA_m.$$

Hence,

$$q = \frac{M_t}{2A_m}.$$

Since,

$$q = \tau_z(s)\delta(s).$$

We get the first Bredt formula:

$$\tau_z(s) = \frac{M_t}{2A_m\delta(s)}.$$

This gives the shear stress in a thin-walled closed section under torsion.

This states that the stress becomes larger when the torque M_t is larger, when the enclosed area A_m is smaller or when the wall thickness $\delta(s)$ is smaller.

5.5.6 The Second Bredt Formula

From Stokes' theorem:

$$\int_{\partial A_m} \tau_z\,\mathrm{d}s = \int_{A_m} \operatorname{curl}\boldsymbol{\tau}_3\cdot\boldsymbol{e}_3\,\mathrm{d}A.$$

But

$$\operatorname{curl}\tau_3\cdot\boldsymbol{e}_3 = 2G\theta'.$$

Therefore,

$$\int_{\partial A_m} \tau_z(s)\,\mathrm{d}s = 2G\theta' A_m.$$

Now, we use the first Bredt formula to obtain:

$$\int_{\partial A_m} \tau_z(s)\,\mathrm{d}s = \int_{\partial A_m} \frac{M_t}{2A_m\delta(s)}\,\mathrm{d}s.$$

Since M_t and A_m are constants,

$$\int_{\partial A_m} \tau_z(s)\,\mathrm{d}s = \frac{M_t}{2A_m}\int_{\partial A_m} \frac{\mathrm{d}s}{\delta(s)}.$$

Equating these two expressions gives:

$$2G\theta' A_m = \frac{M_t}{2A_m}\int_{\partial A_m} \frac{\mathrm{d}s}{\delta(s)}.$$

Solving for M_t/θ', we get the torsional rigidity:

$$C_T = \frac{M_t}{\theta'} = \frac{4GA_m^2}{\int_{\partial A_m} \frac{\mathrm{d}s}{\delta(s)}}.$$

This is the second Bredt formula. This states that torsional rigidity increases with larger shear modulus G, larger enclosed mid-line area A_m and larger wall thickness $\delta(s)$.

If the wall thickness is constant $\delta(s) = \delta$, then

$$\int_{\partial A_m} \frac{\mathrm{d}s}{\delta} = \frac{L}{\delta}.$$

So

$$C_T = \frac{4GA_m^2\delta}{L}.$$

Lecture 13: Plane Stress and Plain Strain 4. Maj 2026

6 Two-Dimensional Elasticity

An elasticity problem may be reduced from three to two dimensions if there is no traction on one plane passing through the body. This state is known as *plane stress* since all nonzero stresses are confined to planes parallel to the traction-free plane.

6.1 Plane Stress Equations

For plane stress with the z-axis stress-free, we have

$$\sigma_{xz} = \sigma_{yz} = \sigma_{zz} = 0.$$

Also, we assume that the remaining stresses do not vary with z but are only functions of x and y. This gives:

$$\begin{aligned} \sigma_{xx,x} + \sigma_{xy,y} + f_x &= 0 \\ \sigma_{yx,x} + \sigma_{yy,y} + f_y &= 0 \\ f_z &= 0 \\ &. \end{aligned}$$

For some linear problems, it is convenient to specify the body force vector in the form

$$\boldsymbol{f} = -\nabla V = -V_{,i\boldsymbol{e}_i}$$

where V is a potential function. This corresponds to a body force field that is conservative. Combining the above equations yiel

$$\begin{aligned} \sigma_{xx,x} + \sigma_{xy,y} - V_{,x} &= 0 \\ \sigma_{xy,x} + \sigma_{yy,y} - V_{,y} &= 0 \\ V_{,z} &= 0. \end{aligned}$$

We can write the stresses as:

$$\begin{aligned}\sigma_{xx} &= V + \phi_{,yy} \\ \sigma_{yy} &= V + \phi_{,xx} \\ \sigma_{xy} &= -\phi_{,xy}\end{aligned}$$

where ϕ is the Airy stress function. Substituting these Airy-forms into the equations above solves them identically.

The constitutive equations are also expressed in terms of ϕ. First the generalized Hooke's law for the plane stress case becomes

$$\begin{aligned}\epsilon_{xx} &= \frac{1}{E}\left(\sigma_{\times} - \nu\sigma_{yy}\right) \\ \epsilon_{yy} &= \frac{\nu}{E}\left(\sigma_{yy} - \nu\sigma_{xx}\right) \\ \epsilon_{zz} &= -\frac{\nu}{E}\left(\sigma_{xx} + \sigma_{yy}\right) \\ \epsilon_{xy} &= \frac{1}{2G}\sigma_{xy} \\ \epsilon_{xz} = \epsilon_{yz} &= 0.\end{aligned}$$

From these, we see that the strains do not depend on z, but only on x and y. However the equation for ϵ_{zz} has a nonzero strain in the z-direction indicating that a state of plane stress does *not* imply a corresponding state of plane strain.

It is also convenient to write the stresses in terms of the strains by solving the equations for $\epsilon_{xx}, \epsilon_{yy}$, and ϵ_{xy} for σ_{xx}, σ_{yy}, and σ_{xy}, respectively:

$$\begin{aligned}\sigma_{xx} &= \frac{E}{\left(1-\nu^2\right)}\left(\epsilon_{xx} + \nu\epsilon_{yy}\right) \\ \sigma_{yy} &= \frac{E}{\left(1-\nu^2\right)}\left(\epsilon_{yy} + \nu\epsilon_{xx}\right) \\ \sigma_{xy} &= 2G\epsilon_{xy}.\end{aligned}$$

Introducing the equations from before gives:

$$\begin{aligned}\epsilon_{xx} &= \frac{1}{E}\left(\left(\phi_{,yy} - \nu\phi_{,\times}\right) + (1-\nu)\,V\right) \\ \epsilon_{yy} &= \frac{1}{E}\left(\left(\phi_{,xx} - \nu\phi_{,yy}\right) + (1-\nu)\,V\right) \\ \epsilon_{zz} &= -\frac{\nu}{E}\left(\left(\phi_{,xx} + \phi_{,yy}\right) + 2V\right) \\ \epsilon_{xy} &= -\frac{1}{2G}\phi_{,xy} \\ \epsilon_{xz} = \epsilon_{yz} &= 0.\end{aligned}$$

We may now establish the compatibility equations in terms of the stress function. First, with substitutions of the strains from above, we get:

$$\frac{1}{E}\left(\phi_{,yyyy} - \nu\phi_{,xxyy} + (1-\nu)\,V_{,yy} + \phi_{,xxxx} - \nu\phi_{,xxyy} + (1-\nu)\,V_{,xx}\right) = -\frac{1}{G}\phi_{,xxyy}.$$

With $E/G = 2(1+\nu)$, this reduces to

$$\nabla^4\phi = -(1-\nu)\,\nabla^2 V.$$

Where the operator $\nabla^4(\dots)$ is defined as:

$$\nabla^4(\dots) = (\dots)_{,xxxx} + 2(\dots)_{,xxyy} + (\dots)_{,yyyy}.$$

We then get:

$$\begin{aligned}
\epsilon_{zz,yy} &= \phi_{,xxyy} + \phi_{,yyyy} + 2V_{,yy} = 0 \\
\epsilon_{zz,xx} &= \phi_{,xxxx} + \phi_{,yyxx} + 2V_{,xx} = 0 \\
\epsilon_{zz,xy} &= \phi_{,xxxy} + \phi_{,yyyx} + 2V_{,xy} = 0.
\end{aligned}$$

The above three equations are normally not considered in elementary plane stress theory, there only $\nabla^4\phi = -(1-\nu)\nabla^2 V$ is used instead. All of the above three equations contain terms related to ϵ_{zz} which is just a Poison effect and therefore very small (i.e. negligible especially for thin members).

6.2 Plane Strain Equations

For plane strain with the z-axis strain-free, we have

$$\epsilon_{xz} = \epsilon_{yz} = \epsilon_{zz} = 0.$$

As in the case of plane stress, all tractions and body forces are functions of x and y only.

To enforce these conditions, we use the generalized Hooke's law:

$$\begin{aligned}
\epsilon_{xx} &= \frac{1}{E}\left(\sigma_{xx} - \nu\left(\sigma_{yy} + \sigma_{zz}\right)\right) \\
\epsilon_{yy} &= \frac{1}{E}\left(\sigma_{yy} - \nu\left(\sigma_{xx} + \sigma_{zz}\right)\right) \\
0 &= \frac{1}{E}\left(\sigma_{zz} - \nu\left(\sigma_{xx} + \sigma_{yy}\right)\right) \\
\epsilon_{xy} &= \frac{1}{2G}\sigma_{xy} \\
\epsilon_{xz} &= \epsilon_{yz} = 0.
\end{aligned}$$

From this, we see that

$$\sigma_{zz} = \nu\left(\sigma_{xx} + \sigma_{yy}\right)$$

so that the strain-free plane is not stress-free, in general. We have

$$\begin{aligned}
\sigma_{xx,x} + \sigma_{xy,y} - V_{,x} &= 0 \\
\sigma_{xy,x} + \sigma_{yy,y} - V_{,y} &= 0 \\
\sigma_{zz} &= \text{constant.}
\end{aligned}$$

Introducing the Airy stress function, the above becomes

$$\begin{aligned}
\epsilon_{xx} &= \frac{1}{E}\left(\left(1-\nu^2\right)\phi_{,yy} - \nu(1+\nu)\phi_{,xx} + (1-\nu(1+2\nu))V\right) \\
\epsilon_{yy} &= \frac{1}{E}\left(\left(1-\nu^2\right)\phi_{,xx} - \nu(1+\nu)\phi_{,yy} + (1-\nu(1+2\nu))V\right) \\
\epsilon_{xy} &= -\frac{1}{2G}\phi_{,xy}.
\end{aligned}$$

Then we rewrite

$$\epsilon_{xx,yy} + \epsilon_{yy,xx} = 2\epsilon_{xy,xy}$$

using the above to get

$$\nabla^4\phi = -\frac{1-2\nu}{1-\nu}\nabla^2 V.$$

The resulting homogeneous compatibility equation:

$$\nabla^4\phi = 0$$

is known as the biharmonic equation.